MIRROR EXTENSIONS OF VERTEX OPERATOR ALGEBRAS

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Abstract. The mirror extensions for vertex operator algebras are studied. Two explicit examples which are not simple current extensions of some affine vertex operator algebras of type A are given.

1. Introduction

Mirror extensions, in the title of this paper, refer to a general Theorem 3.8 in [X2] which produces completely rational conformal nets from given ones. Based on the close relations between conformal nets and vertex operator algebras, we make the following conjecture which is the vertex operator algebra version of Theorem 3.8 in [X2]:

Mirror Extension Conjecture. *Let V be a rational and C_2-cofinite vertex operator algebra and U a rational and C_2-cofinite vertex operator subalgebra of V. Denote U^c the commutant vertex operator algebra of U in V. Assume that $(U^c)^c = U$, and*

$$V = U \otimes U^c \bigoplus (\oplus_{i=1}^n U_i \otimes U_i^c),$$

where U_i's and U_i^c's are irreducible modules for U and U^c respectively. Then if

$$U^e = U \bigoplus (\oplus_{i=1}^n m_i U_i)$$

is a rational vertex operator algebra where $m_i \geq 0$, so is

$$(U^c)^e = U^c \bigoplus (\oplus_{i=1}^n m_i U_i^c).$$

The mirror extensions of conformal nets associated to affine Kac-Moody algebras of type A have been studied extensively in [X2]. It is conjectured on P.846 of [X2] that there should be rational vertex operator algebras corresponding to the class of completely rational conformal nets constructed in §4.3 of [X2], and this conjecture which is a special case of Mirror Extension Conjecture is the motivation of our paper.

A proof of these conjectures seems to be out of reach at present. Instead we focus on two interesting examples of mirror extensions of vertex operator algebras (cf. P.836 of [X2]) in this paper. The first example is based on conformal inclusions $SU(2)_{10} \subset Spin(5)_1$ and $SU(2)_{10} \times SU(10)_2 \subset SU(20)_1$. The spectrum of $SU(2)_{10} \subset Spin(5)_1$ is $H_0 + H_6$, and $(6, \Lambda_3 + \Lambda_7)$ appears in the spectrum of $SU(2)_{10} \times SU(10)_2 \subset SU(20)_1$. Here we use Λ_i to denote the fundamental weights of $SU(n)$, and 0 (or Λ_0) to denote the trivial representation of $SU(n)$ and we specialize the case $SU(2)$ by using i to denote the highest weight of the representation of $SU(2)$. Theorem 3.8 in [X2] implies that there exists a completely rational net containing $\mathcal{A}_{SU(10)_2}$ with spectrum $H_0 + H_{\Lambda_3+\Lambda_7}$.

From vertex operator algebra point of view, this suggests that there should be a vertex operator algebra structure on $L_{\mathfrak{sl}(10)}(2, 0) + L_{\mathfrak{sl}(10)}(2, \Lambda_3 + \Lambda_7)$ where $L_{\mathfrak{g}}(k, \lambda)$ is the highest weight irreducible module for the affine Lie algebra $\hat{\mathfrak{g}}$ of level k associated to the weight λ of

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$\mathfrak{g}$. Notice that the lowest weight $h_{\Lambda_3+\Lambda_7}$ of $L_{\mathfrak{sl}(10)}(2,\Lambda_3+\Lambda_7)$ is 2 and $L_{\mathfrak{sl}(10)}(2,\Lambda_3+\Lambda_7)$ is not a simple current, such a vertex operator algebra has not been obtained from the affine vertex operator algebra in the literature. We need to find intertwining operators associated to vectors in $L_{\mathfrak{sl}(10)}(2,\Lambda_3+\Lambda_7)$ verifying the crucial locality condition of vertex operator algebra. The correlation functions of such intertwining operators are solutions of KZ equation, and locality means that such functions are symmetric rational functions. Thus in this case we need to find symmetric rational solutions to KZ equation for $SU(10)_2$.

There are contour integral representations of solutions to KZ equation for $SU(n)_k$ (cf. [FEK] and references therein) for generic level $k \notin \mathbb{Q}$. It is not clear how to find general solutions and pick out a particular rational solution in our case with $n = 10$ and $k = 2$. Instead we take a different approach, which contains the key idea of this paper. First we note that from the conformal inclusion $SU(2)_{10} \subset Spin(5)$, the primary fields in $L_{\mathfrak{sl}(2)}(10,6)$ in this inclusion produce correlator X which is a symmetric rational solution of KZ equation for the affine vertex operator algebra $L_{\mathfrak{sl}(2)}(10,0)$. This is equivalent to say that these solutions are invariant under braiding operator B. By abusing of notation we write $BX = X$. Note that $(6,\Lambda_3+\Lambda_7)$ appears in the spectrum of $SU(2)_{10} \times SU(10)_2 \subset SU(20)_1$, the vertex operator associated to the highest weight vector of $(6,\Lambda_3+\Lambda_7)$ will give us a symmetric rational function. Due to a crucial non-degenerate property in Corollary 2.13, this implies $B'\dot{B} = Id$, where B' is similar to B by conjugation of invertible diagonal matrix. Due to the crossing symmetry property of B for $SU(2)$ in Lemma 2.7 from $BX = X$ we conclude there must be $\dot{X}$ which verifies KZ equation for $L_{\mathfrak{sl}(10)}(2,0)$ such that $\dot{B}\dot{X} = \dot{X}$, and it follows that $\dot{X}$ is a symmetric rational function.

From $\dot{X}$ we can easily define a vertex operator algebra structure on

$$L_{\mathfrak{sl}(10)}(2,0) \oplus L_{\mathfrak{sl}(10)}(2,\Lambda_3+\Lambda_7),$$

and we derive our main result Theorem 3.1. The vertex operator algebra in Theorem 3.1 is an example of mirror extension, which is constructed by an idea very different from what is previously known. We are informed recently that the vertex operator algebra $L_{\mathfrak{sl}(10)}(2,0) \oplus L_{\mathfrak{sl}(10)}(2,\Lambda_3+\Lambda_7)$ can also be realized a coset construction in a holomorphic vertex operator algebra with central charge 24 [L].

The second example is based on the conformal inclusion $SU(2)_{28} \subset (G_2)_1$ (see [CIZ, GNO]) and the level-rank duality $SU(2)_{28} \times SU(28)_2 \subset SU(56)_1$. Similar to the first example,

$$L_{\mathfrak{sl}(28)}(2,0) \oplus L_{\mathfrak{sl}(28)}(2,\Lambda_5+\Lambda_{23}) \oplus L_{\mathfrak{sl}(28)}(2,\Lambda_9+\Lambda_{19}) \oplus L_{\mathfrak{sl}(28)}(2,2\Lambda_{14})$$

is a vertex operator algebra which is a mirror extension corresponding to the vertex operator algebra

$$L_{G_2}(1,0) = L_{\mathfrak{sl}(2)}(28,0) \oplus L_{\mathfrak{sl}(2)}(28,10) \oplus L_{\mathfrak{sl}(2)}(28,18) \oplus L_{\mathfrak{sl}(2)}(28,28).$$

Although these two examples of mirror extensions which are not simple current extensions are totally new in the theory of vertex operator algebra, the mirror extension, in fact, is a general phenomenon. Many well known vertex operator algebras in the literature can also be regraded as mirror extensions. We give two easy examples here. The first example comes from the well known GKO-construction [GKO]:

$$\begin{aligned} L_{\mathfrak{sl}(2)}(1,0) \otimes L_{\mathfrak{sl}(2)}(3,0) = L(\frac{4}{5},0) \otimes L_{\mathfrak{sl}(2)}(4,0) \oplus L(\frac{4}{5},\frac{2}{3}) \otimes L_{\mathfrak{sl}(2)}(4,2) \\ \oplus L(\frac{4}{5},3) \otimes L_{\mathfrak{sl}(2)}(4,4), \end{aligned}$$

where $L(\frac{4}{5},h)$ is the lowest weight irreducible module for the Virasoro algebra with central charge $\frac{4}{5}$ and lowest weight h. The vertex operator algebra structure on $L(\frac{4}{5},0) \oplus L(\frac{4}{5},3)$ is well known

now (see [KMY]), which is a simple current extension of $L(\frac{4}{5},0)$. The vertex operator algebra $L_{\mathfrak{sl}(2)}(4,0)\oplus L_{\mathfrak{sl}(2)}(4,4)$ (see [MS, Li3]) is a mirror extension.

The other is the $3A$ algebra U [LYY] which has a decomposition:

$$\begin{aligned} U \cong & L(\frac{4}{5},0)\otimes L(\frac{6}{7},0)\oplus L(\frac{4}{5},3)\otimes L(\frac{6}{7},5)\oplus L(\frac{4}{5},\frac{2}{3})\otimes L(\frac{6}{7},\frac{4}{3}) \\ & \oplus L(\frac{4}{5},\frac{13}{8})\otimes L(\frac{6}{7},\frac{3}{8})\oplus L(\frac{4}{5},\frac{1}{8})\otimes L(\frac{6}{7},\frac{23}{8}). \end{aligned}$$

Again the vertex operator algebra $L(\frac{6}{7},0)\oplus L(\frac{6}{7},5)$ [LY] is a mirror extension corresponding to the vertex operator algebra $L(\frac{4}{5},0)\oplus L(\frac{4}{5},3)$.

Besides what is already described above, we have included a preliminary section §2 on affine vertes operator algebras, KZ equation, primary fields, conformal nets and induction, and we prove the crucial non-degeneracy condition in Corollary 2.13. The first nontrivial example of mirror extensions is presented in §3. The uniqueness of this vertex operator algebra structure is obtained in §4. In §5 we construct another example by using similar method. In §6 we discuss problems about general case. The last section is the appendix which is devoted to proving the non-degeneracy property given in Corollary 2.13 using vertex operator algebra language.

2. Preliminaries

2.1. **Preliminaries on sectors.** Given an infinite factor M, the *sectors of* M are given by

$$\mathrm{Sect}(M)=\mathrm{End}(M)/\mathrm{Inn}(M),$$

namely $\mathrm{Sect}(M)$ is the quotient of the semigroup of the endomorphisms of M modulo the equivalence relation: $\rho,\rho'\in\mathrm{End}(M)$, $\rho\sim\rho'$ iff there is a unitary $u\in M$ such that $\rho'(x)=u\rho(x)u^*$ for all $x\in M$.

$\mathrm{Sect}(M)$ is a *-semiring (there are an addition, a product and an involution $\rho\to\bar\rho$) equivalent to the Connes correspondences (bimodules) on M up to unitary equivalence. If ρ is an element of $\mathrm{End}(M)$ we shall denote by $[\rho]$ its class in $\mathrm{Sect}(M)$. We define $\mathrm{Hom}(\rho,\rho')$ between the objects $\rho,\rho'\in\mathrm{End}(M)$ by

$$\mathrm{Hom}(\rho,\rho')\equiv\{a\in M: a\rho(x)=\rho'(x)a\ \forall x\in M\}.$$

We use $\langle\lambda,\mu\rangle$ to denote the dimension of $\mathrm{Hom}(\lambda,\mu)$; it can be ∞, but it is finite if λ,μ have finite index. See [J] for the definition of index for type II_1 case which initiated the subject and [PP] for the definition of index in general. Also see §2.3 [KLX] for expositions. $\langle\lambda,\mu\rangle$ depends only on $[\lambda]$ and $[\mu]$. Moreover we have if ν has finite index, then $\langle\nu\lambda,\mu\rangle=\langle\lambda,\bar\nu\mu\rangle$, $\langle\lambda\nu,\mu\rangle=\langle\lambda,\mu\bar\nu\rangle$ which follows from Frobenius duality. μ is a subsector of λ if there is an isometry $v\in M$ such that $\mu(x)=v^*\lambda(x)v,\forall x\in M$. We will also use the following notation: if μ is a subsector of λ, we will write as $\mu\prec\lambda$ or $\lambda\succ\mu$. A sector is said to be irreducible if it has only one subsector.

2.2. **Local nets.** By an interval of the circle we mean an open connected non-empty subset I of S^1 such that the interior of its complement I' is not empty. We denote by $\mathcal{I}$ the family of all intervals of S^1. A *net* $\mathcal{A}$ of von Neumann algebras on S^1 is a map

$$I\in\mathcal{I}\to\mathcal{A}(I)\subset B(\mathcal{H})$$

from $\mathcal{I}$ to von Neumann algebras on a fixed separable Hilbert space $\mathcal{H}$ that satisfies:

A. *Isotony.* If $I_1\subset I_2$ belong to $\mathcal{I}$, then

$$\mathcal{A}(I_1)\subset\mathcal{A}(I_2).$$

If $E\subset S^1$ is any region, we shall put $\mathcal{A}(E)\equiv\bigvee_{E\supset I\in\mathcal{I}}\mathcal{A}(I)$ with $\mathcal{A}(E)=\mathbb{C}$ if E has empty interior (the symbol $\vee$ denotes the von Neumann algebra generated).

The net $\mathcal{A}$ is called *local* if it satisfies:

B. *Locality.* If $I_1, I_2 \in \mathcal{I}$ and $I_1 \cap I_2 = \emptyset$ then

$$[\mathcal{A}(I_1), \mathcal{A}(I_2)] = \{0\},$$

where brackets denote the commutator.

The net $\mathcal{A}$ is called *Möbius covariant* if in addition satisfying the following properties **C,D,E,F**:

C. *Möbius covariance.* There exists a non-trivial strongly continuous unitary representation U of the Möbius group Möb (isomorphic to $PSU(1,1)$) on $\mathcal{H}$ such that

$$U(g)\mathcal{A}(I)U(g)^* \;=\; \mathcal{A}(gI), \quad g \in \text{Möb},\ I \in \mathcal{I}.$$

D. *Positivity of the energy.* The generator of the one-parameter rotation subgroup of U (conformal Hamiltonian), denoted by L_0 in the following, is positive.

E. *Existence of the vacuum.* There exists a unit U-invariant vector $\Omega \in \mathcal{H}$ (vacuum vector), and Ω is cyclic for the von Neumann algebra $\bigvee_{I\in\mathcal{I}} \mathcal{A}(I)$.

By the Reeh-Schlieder theorem Ω is cyclic and separating for every fixed $\mathcal{A}(I)$. The modular objects associated with $(\mathcal{A}(I), \Omega)$ have a geometric meaning

$$\Delta_I^{it} = U(\Lambda_I(2\pi t)), \qquad J_I = U(r_I)\ .$$

Here Λ_I is a canonical one-parameter subgroup of Möb and $U(r_I)$ is an antiunitary acting geometrically on $\mathcal{A}$ as a reflection r_I on S^1.

This implies *Haag duality*:

$$\mathcal{A}(I)' = \mathcal{A}(I'), \quad I \in \mathcal{I}\ ,$$

where I' is the interior of $S^1 \setminus I$.

F. *Irreducibility.* $\bigvee_{I\in\mathcal{I}} \mathcal{A}(I) = B(\mathcal{H})$. Indeed $\mathcal{A}$ is irreducible iff Ω is the unique U-invariant vector (up to scalar multiples). Also $\mathcal{A}$ is irreducible iff the local von Neumann algebras $\mathcal{A}(I)$ are factors. In this case they are either $\mathbb{C}$ or III_1-factors with separable predual in Connes classification of type III factors.

By a *conformal net* (or diffeomorphism covariant net) $\mathcal{A}$ we shall mean a Möbius covariant net such that the following holds:

G. *Conformal covariance.* There exists a projective unitary representation U of $\mathrm{Diff}(S^1)$ on $\mathcal{H}$ extending the unitary representation of Möb such that for all $I \in \mathcal{I}$ we have

$$\begin{aligned} U(\phi)\mathcal{A}(I)U(\phi)^* \;&=\; \mathcal{A}(\phi.I), \quad \phi \in \mathrm{Diff}(S^1), \\ U(\phi)xU(\phi)^* \;&=\; x, \quad x \in \mathcal{A}(I),\ \phi \in \mathrm{Diff}(I'), \end{aligned}$$

where $\mathrm{Diff}(S^1)$ denotes the group of smooth, positively oriented diffeomorphism of S^1 and $\mathrm{Diff}(I)$ the subgroup of diffeomorphisms g such that $\phi(z) = z$ for all $z \in I'$.

A (DHR) representation π of $\mathcal{A}$ on a Hilbert space $\mathcal{H}$ is a map $I \in \mathcal{I} \mapsto \pi_I$ that associates to each I a normal representation of $\mathcal{A}(I)$ on $B(\mathcal{H})$ such that

$$\pi_{\tilde{I}}|\mathcal{A}(I) = \pi_I, \quad I \subset \tilde{I}, \quad I, \tilde{I} \subset \mathcal{I}\ .$$

π is said to be Möbius (resp. diffeomorphism) covariant if there is a projective unitary representation U_π of Möb (resp. $\mathrm{Diff}(S^1)$) on $\mathcal{H}$ such that

$$\pi_{gI}(U(g)xU(g)^*) = U_\pi(g)\pi_I(x)U_\pi(g)^*$$

for all $I \in \mathcal{I}$, $x \in \mathcal{A}(I)$ and $g \in$ Möb (resp. $g \in \mathrm{Diff}(S^1)$).

By definition the irreducible conformal net is in fact an irreducible representation of itself and we will call this representation the *vacuum representation*.

Let G be a simply connected compact Lie group. By Th. 3.2 of [FG], the vacuum positive energy representation of the loop group LG (cf. [PS]) at level k gives rise to an irreducible conformal net denoted by $\mathcal{A}_{G_k}$. By Th. 3.3 of [FG], every irreducible positive energy representation of the loop group LG at level k gives rise to an irreducible covariant representation of $\mathcal{A}_{G_k}$.

Given an interval I and a representation π of $\mathcal{A}$, there is an *endomorphism of $\mathcal{A}$ localized in* I equivalent to π; namely ρ is a representation of $\mathcal{A}$ on the vacuum Hilbert space $\mathcal{H}$, unitarily equivalent to π, such that $\rho_{I'} = \mathrm{id} \restriction \mathcal{A}(I')$. We now define the statistics. Given the endomorphism ρ of $\mathcal{A}$ localized in $I \in \mathcal{I}$, choose an equivalent endomorphism ρ_0 localized in an interval $I_0 \in \mathcal{I}$ with $\bar{I}_0 \cap \bar{I} = \emptyset$ and let u be a local intertwiner in $\mathrm{Hom}(\rho, \rho_0)$, namely $u \in \mathrm{Hom}(\rho_{\tilde{I}}, \rho_{0,\tilde{I}})$ with I_0 following clockwise I inside $\tilde{I}$ which is an interval containing both I and I_0.

The *statistics operator* $\epsilon(\rho, \rho) := u^* \rho(u) = u^* \rho_{\tilde{I}}(u)$ belongs to $\mathrm{Hom}(\rho^2_{\tilde{I}}, \rho^2_{\tilde{I}})$. We will call $\epsilon(\rho, \rho)$ the positive or right braiding and $\tilde{\epsilon}(\rho, \rho) := \epsilon(\rho, \rho)^*$ the negative or left braiding.

Let $\mathcal{B}$ be a conformal net. By a *conformal subnet* (cf. [Lo]) we shall mean a map

$$I \in \mathcal{I} \to \mathcal{A}(I) \subset \mathcal{B}(I)$$

that associates to each interval $I \in \mathcal{I}$ a von Neumann subalgebra $\mathcal{A}(I)$ of $\mathcal{B}(I)$, which is isotonic

$$\mathcal{A}(I_1) \subset \mathcal{A}(I_2), I_1 \subset I_2,$$

and conformal covariant with respect to the representation U, namely

$$U(g)\mathcal{A}(I)U(g)^* = \mathcal{A}(g.I)$$

for all $g \in \mathrm{Diff}(S^1)$ and $I \in \mathcal{I}$. Note that by Lemma 13 of [Lo] for each $I \in \mathcal{I}$ there exists a conditional expectation $E_I : \mathcal{B}(I) \to \mathcal{A}(I)$ such that E_I preserves the vector state given by the vacuum of $\mathcal{A}$.

Definition 2.1. *Let $\mathcal{A}$ be a conformal net. A conformal net $\mathcal{B}$ on a Hilbert space $\mathcal{H}$ is an extension of $\mathcal{A}$ or $\mathcal{A}$ is a subnet of $\mathcal{B}$ if there is a DHR representation π of $\mathcal{A}$ on $\mathcal{H}$ such that $\pi(\mathcal{A}) \subset \mathcal{B}$ is a conformal subnet. The extension is irreducible if $\pi(\mathcal{A}(I))' \cap \mathcal{B}(I) = \mathbb{C}$ for some (and hence all) interval I, and is of finite index if $\pi(\mathcal{A}(I)) \subset \mathcal{B}(I)$ has finite index for some (and hence all) interval I. The index will be called the index of the inclusion $\pi(\mathcal{A}) \subset \mathcal{B}$ and is denoted by $[\mathcal{B} : \mathcal{A}]$. If π as representation of $\mathcal{A}$ decomposes as $[\pi] = \sum_\lambda m_\lambda[\lambda]$ where m_λ are non-negative integers and λ are irreducible DHR representations of $\mathcal{A}$, we say that $[\pi] = \sum_\lambda m_\lambda[\lambda]$ is the spectrum of the extension. For simplicity we will write $\pi(\mathcal{A}) \subset \mathcal{B}$ simply as $\mathcal{A} \subset \mathcal{B}$.*

2.3. **Induction.** Let $\mathcal{B}$ be a conformal net and $\mathcal{A}$ a subnet. We assume that $\mathcal{A}$ is strongly additive and $\mathcal{A} \subset \mathcal{B}$ has finite index. Fix an interval $I_0 \in \mathcal{I}$ and canonical endomorphism (cf. [LR]) γ associated with $\mathcal{A}(I_0) \subset \mathcal{B}(I_0)$. Given a DHR endomorphism ρ of $\mathcal{B}$ localized in I_0, the α-induction α_ρ of ρ is the endomorphism of $\mathcal{B}(I_0)$ given by

$$\alpha_\rho \equiv \gamma^{-1} \cdot \mathrm{Ad}\epsilon(\rho, \lambda) \cdot \rho \cdot \gamma$$

where ϵ denotes the right braiding (cf. [X2]).

Note that $\mathrm{Hom}(\alpha_\lambda, \alpha_\mu) =: \{x \in \mathcal{B}(I_0) | x\alpha_\lambda(y) = \alpha_\mu(y)x, \forall y \in \mathcal{B}(I_0)\}$ and $\mathrm{Hom}(\lambda, \mu) =: \{x \in \mathcal{A}(I_0) | x\lambda(y) = \mu(y)x, \forall y \in \mathcal{A}(I_0)\}$.

2.4. **Preliminaries on VOAs.** We first recall some basic notions from [FLM, Z, DLM1]. Let $V = \oplus_{n \geq 0} V_n$ be a vertex operator algebra as defined in [FLM] (see also [B]). V is called of *CFT* type if $\mathrm{dim} V_0 = 1$. A *weak V-module M* is a vector space equipped with a linear map

$$\begin{aligned} Y_M(\cdot, z) : & V \to (\mathrm{End} M)[[z, z^1]] \\ & v \mapsto Y_M(v, z) = \sum_{n \in \mathbb{Z}} v_n z^{-n-1} \ (v_n \in \mathrm{End} M) \end{aligned}$$

which satisfies the following conditions for $u \in V$, $v \in V$, $w \in M$ and $n \in \mathbb{Z}$,

$$u_n w = 0 \text{ for } n >> 0;$$

$$Y_M(\mathbf{1}, z) = \mathrm{id}_M;$$

$$z_0^{-1}\delta(\frac{z_1 - z_2}{z_0})Y_M(u, z_1)Y_M(v, z_2) - z_0^{-1}\delta(\frac{z_2 - z_1}{-z_0})Y_M(v, z_2)Y_M(u, z_1)$$
$$= z_2^{-1}\delta(\frac{z_1 - z_0}{z_2})Y_M(Y(u, z_0)v, z_2).$$

A *(ordinary) V-module* is a weak V-module M which carries a $\mathbb{C}$-grading induced by the spectrum of $L(0)$ where $L(0)$ is a component operator of

$$Y_M(\omega, z) = \sum_{n\in\mathbb{Z}} L(n)z^{-n-2}.$$

That is, $M = \oplus_{\lambda\in\mathbb{C}} M_\lambda$ where $M_\lambda = \{w \in M | L(0)w = \lambda w\}$. Moreover one requires that M_λ is finite dimensional and for fixed λ, $M_{n+\lambda} = 0$ for all small enough integers n. An *admissible* V-module is a weak V-module M which carries a $\mathbb{Z}_+$-grading $M = \oplus_{n\in\mathbb{Z}_+} M(n)$ that satisfies the following

$$v_m M(n) \subset M(n + \mathrm{wt}v - m - 1)$$

for homogeneous $v \in V$. It is easy to show that any *ordinary* module is *admissible*. And for an *admissible* V-module $M = \bigoplus_{n\in\mathbb{Z}_+} M(n)$, the contragredient module M' is defined in [FHL] as follows:

$$M' = \bigoplus_{n\in\mathbb{Z}_+} M(n)^*,$$

where $M(n)^* = \mathrm{Hom}_{\mathbb{C}}(M(n), \mathbb{C})$. The vertex operator $Y_{M'}(a, z)$ is defined for $a \in V$ via

$$\langle Y_{M'}(a, z)f, u\rangle = \langle f, Y_M(e^{zL(1)}(-z^{-2})^{L(0)}a, z^{-1})u\rangle,$$

where $\langle f, w\rangle = f(w)$ is the natural paring $M' \times M \to \mathbb{C}$. V is called *rational* if every admissible V-module is completely reducible. It is proved in [DLM2] that if V is rational then there are only finitely many irreducible admissible V-modules up to isomorphism and each irreducible admissible V-module is ordinary. Let $M^0, \cdots, M^p$ be the irreducible modules up to isomorphism with $M^0 = V$. Then there exist $h_i \in \mathbb{C}$ for $i = 0, \cdots, p$ such that

$$M^i = \oplus_{n=0}^{\infty} M^i_{h_i+n}$$

where $M^i_{h_i} \neq 0$ and $L(0)|_{M^i_{h_i+n}} = h_i + n$, $\forall n \in \mathbb{Z}_+$. h_i is called the *conformal weight* of M^i. We denote $M^i(n) = M^i_{h_i+n}$. Moreover, h_i and the central charge c are rational numbers (see [DLM3]). Let $h_{\min}$ be the minimum of h_i's. The effective central charge $\tilde{c}$ is defined as $c-24h_{\min}$. For each M^i we define the q-character of M^i by

$$\mathrm{ch}_q M^i = q^{-c/24} \sum_{n\geq 0} (\dim M^i_{h_i+n})q^{h_i+n}.$$

V is called *C_2-cofinite* if $\dim V/C_2(V) < \infty$ where $C_2(V) = \langle u_{-2}v | u, v \in V\rangle$ [Z]. Rationality and C_2-cofiniteness are two important concepts in the theory of vertex operator algebras as most good results in the field need both assumptions.

Take a formal power series in q or a complex function $f(z) = q^h \sum_{n\geq 0} a_n q^n$. We say that the coefficients of $f(q)$ satisfy the *polynomial growth condition* if there exist positive numbers A and α such that $|a_n| \leq An^\alpha$.

If V is rational and C_2-cofinite, then $\mathrm{ch}_q M^i$ converges to a holomorphic function on the upper half plane [Z]. Using the modular invariance result from [Z] and results on vector valued modular forms from [KM] we have (see [DM])

Lemma 2.2. *Let V be rational and C_2-cofinite. For each i, the coefficients of $\eta(q)^{\tilde{c}}\mathrm{ch}_q M^i$ satisfy the polynomial growth condition where $\eta(q) = q^{1/24}\prod_{n\geq 1}(1-q^n)$.*

Definition 2.3. *Let V be a vertex operator algebra and let (M^i, Y_i), (M^j, Y_j), (M^k, Y_k) be three V-modules. An intertwining operator of type $\left(\begin{array}{c} M^k \\ M^i \ \ M^j \end{array}\right)$ is a linear map*

$$\begin{aligned} \mathcal{Y}(\cdot, z) : M^i \quad & \to \ (Hom(M^j, M^k))\{z\} \\ v \in M^i \quad & \mapsto \ \mathcal{Y}(v, z) \in (Hom(M^j, M^k))\{z\} \end{aligned}$$

satisfying the following axioms:

1. For any $u \in M^i$,

$$\mathcal{Y}(L(-1)u, z) = \frac{d}{dz}\mathcal{Y}(u, z),$$

where $L(n)$ is the component operator of $Y_i(w, z) = \sum_{n\in\mathbb{Z}} L(n)z^{-n-2}$;

2. $\forall u \in V,\ v \in M^i$,

$$\begin{aligned} & z_0^{-1}\delta(\tfrac{z_1-z_2}{z_0})Y_k(u, z_1)\mathcal{Y}(v, z_2) - z_0^{-1}\delta(\tfrac{z_2-z_1}{-z_0})\mathcal{Y}(v, z_2)Y_j(u, z_1) \\ & \qquad = z_2^{-1}\delta(\tfrac{z_1-z_0}{z_2})\mathcal{Y}(Y_i(u, z_0)v, z_2). \end{aligned}$$

The intertwining operators of type $\left(\begin{array}{c} M^k \\ M^i \ \ M^j \end{array}\right)$ form a vector space denoted by $\mathcal{V}_{i,j}^k$. The dimension of this vector space is called the *fusion rule* for M^i, M^j and M^k, and is denoted by $N_{i,j}^k$. We will use $\mathcal{Y}_{i,j}^k$ to denote an intertwining operator in $\mathcal{V}_{i,j}^k$. Assume that $M^s = \sum_{n\in\mathbb{Z}_+} M^s_{\lambda_s+n}$ for $s = i, j, k$. Then for any $\mathcal{Y} \in \mathcal{V}_{i,j}^k$, we know from [FHL] that for $u \in M^i$ and $v \in M^j$

$$\mathcal{Y}(u, z)v \in z^{\Delta(\mathcal{Y})}M^k[[z, z^{-1}]],$$

where $\Delta(\mathcal{Y}) = \lambda_k - \lambda_i - \lambda_j$.

We now turn our discussion to four point functions (correlation functions). Let V be a rational and C_2-cofinite vertex operator algebra of CFT type and $V \cong V'$. By Lemma 4.1 in [H2], one knows that for $u_{a_i} \in M^{a_i}$,

$$\langle u_{a_4'}, \mathcal{Y}_{a_1,a_5}^{a_4}(u_{a_1}, z_1)\mathcal{Y}_{a_2,a_3}^{a_5}(u_{a_2}, z_2)u_{a_3}\rangle,$$

$$\langle u_{a_4'}, \mathcal{Y}_{a_2,a_6}^{a_4}(u_{a_2}, z_2)\mathcal{Y}_{a_1,a_3}^{a_6}(u_{a_1}, z_1)u_{a_3}\rangle,$$

are analytic on $|z_1| > |z_2| > 0$ and $|z_2| > |z_1| > 0$ respectively and can both be analytically extended to multi-valued analytic functions on

$$R = \{(z_1, z_2) \in \mathbb{C}^2 | z_1, z_2 \neq 0, z_1 \neq z_2\}.$$

We can lift the multi-valued analytic functions on R to single-valued analytic functions on the universal covering $\tilde{R}$ of R as in [H3]. We use

$$E\langle u_{a_4'}, \mathcal{Y}_{a_1,a_5}^{a_4}(u_{a_1}, z_1)\mathcal{Y}_{a_2,a_3}^{a_5}(u_{a_2}, z_2)u_{a_3}\rangle$$

and

$$E\langle u_{a_4'}, \mathcal{Y}_{a_2,a_6}^{a_4}(u_{a_2}, z_2)\mathcal{Y}_{a_1,a_3}^{a_6}(u_{a_1}, z_1)u_{a_3}\rangle$$

to denote those analytic functions.

Let $\{\mathcal{Y}_{a,b;i}^c | i = 1, \cdots, N_{a,b}^c\}$ be a basis of $\mathcal{V}_{a,b}^c$. The linearly independency of

$$\{E\langle u_{a_4'}, \mathcal{Y}_{a_1,a_5;i}^{a_4}(u_{a_1}, z_1)\mathcal{Y}_{a_2,a_3;j}^{a_5}(u_{a_2}, z_2)u_{a_3}\rangle | i = 1, \cdots, N_{a_1,a_5}^{a_4}, j = 1, \cdots, N_{a_2,a_3}^{a_5}, \forall a_5\}$$

follows from [H3].

2.5. **Primary fields for affine VOA and KZ equation.** In this section, we briefly review the construction of the affine vertex operator algebra associated to the integrable highest weight modules for the affine Kac-Moody Lie algebras, and also give the KZ equation [KZ, KT] of the correlation functions.

Definition 2.4. *Let W be a vector space, a weak vertex operator on W is a formal series*

$$a(z) = \sum_{n\in\mathbb{Z}} a_n z^{-n-1} \in (End\ W)[[z, z^{-1}]]$$

such that for every $w \in W$, $a_n w = 0$, for n sufficiently large.

Let $a(z)$ and $b(z)$ be two weak vertex operators on W, define

$$a(z)_n b(z) = \mathrm{Res}_{z_1}((z_1 - z)^n a(z_1)b(z) - (-z + z_1)^n b(z)a(z_1)). \tag{2.1}$$

This is also a weak vertex operator on W (see [LL]).

The following important lemma will be useful later.

Lemma 2.5. [Li2] *If $a(z)$, $b(z)$ and $c(z)$ are pairwise mutually local weak vertex operators, then $a(z)_n b(z)$ and $c(z)$ are mutually local.*

Let $\mathfrak{g}$ be a finite-dimensional simple Lie algebra with a nondegenerate symmetric invariant bilinear form and a Cartan subalgebra $\mathfrak{h}$. Let $\hat{\mathfrak{g}} = \mathbb{C}[t, t^{-1}]\otimes\mathfrak{g}\oplus\mathbb{C}K$ be the corresponding affine Lie algebra. For any $X \in \mathfrak{g}$, set $X(n) = X \otimes t^n$ and $X(z) = \sum_{n\in\mathbb{Z}} X(n)z^{-n-1}$. Fix a positive integer k. Then any $\lambda \in \mathfrak{h}^*$ can be viewed as a linear form on $\mathbb{C}K\oplus\mathfrak{h} \subset \hat{\mathfrak{g}}$ by sending K to k. Let us denote the corresponding irreducible highest weight module for $\hat{\mathfrak{g}}$ associated to a highest weight λ by $L_{\mathfrak{g}}(k, \lambda)$. It is proved that $L_{\mathfrak{g}}(k, 0)$ is a rational vertex operator algebra [DL, FZ, Li2] with all the inequivalent irreducible modules $\{L_{\mathfrak{g}}(k, \lambda)|\langle\lambda, \theta\rangle \le k, \lambda \in \mathfrak{h}^*, \lambda \text{ is an integral dominant weight}\}$, where θ is the longest root of $\mathfrak{g}$ and $(\theta, \theta) = 2$.

$L_{\mathfrak{g}}(k, 0)$ has a basis $\{X_{i_1}(-n_1)\cdots X_{i_t}(-n_t)\mathbf{1}|X_{i_s} \in \mathfrak{g}, n_s \in \mathbb{Z}_+, s = 1, \cdots, t\}$. The vertex operator on $L_{\mathfrak{g}}(k, 0)$ is defined as

$$Y(X(-1)\mathbf{1}, z) = \sum_{n\in\mathbb{Z}} X(n)z^{-n-1};$$
$$Y(X_{i_1}(-n_1)\cdots X_{i_t}(-n_t)\mathbf{1}) = X_{i_1}(z)_{-n_1}\cdots X_{i_t}(z)_{-n_t}\mathbf{1}.$$

Let $d = \dim\mathfrak{g}$, and let $\{u^{(1)}, \cdots, u^{(d)}\}$ be an orthogonal basis of $\mathfrak{g}$ with respect to the bilinear form on $\mathfrak{g}$. Then set

$$\omega = \frac{1}{2(k + \check{h})} \sum_{i=1}^{d} u^{(i)}(-1)u^{(i)}(-1)\mathbf{1},$$

where $\check{h}$ is the dual Coxeter number of $\mathfrak{g}$. Define operators $L(n)$ for $n \in \mathbb{Z}$ by:

$$Y(\omega, z) = \sum_{n\in\mathbb{Z}} L(n)z^{-n-2}.$$

The operators $L(n)$ gives representation of the Virasoro algebra on any $L_{\mathfrak{g}}(k, 0)$-modules with central charge $c = \frac{k\cdot\dim\mathfrak{g}}{2(k+\check{h})}$. Let $\mathcal{Y}(\cdot, z)$ be an intertwining operator of type

$$\left(\begin{array}{c} L_{\mathfrak{g}}(k, \lambda_3) \\ L_{\mathfrak{g}}(k, \lambda_2)\ \ L_{\mathfrak{g}}(k, \lambda_1) \end{array}\right),$$

then by [FZ]

$$\mathcal{Y}(u, z) = \sum_{n\in\mathbb{Z}} u_n z^{-n-1} z^{\Delta(\mathcal{Y})},$$

where $u_n \in \mathrm{Hom}(L_{\mathfrak{g}}(k,\lambda_1), L_{\mathfrak{g}}(k,\lambda_3))$, and $u_n L_{\mathfrak{g}}(k,\lambda_1)(m) \subset L_{\mathfrak{g}}(k,\lambda_3)(m+\mathrm{wt}u-n-1)$, where $\mathrm{wt}u = i$ means that $u \in L_{\mathfrak{g}}(k,\lambda_2)(i)$. The following commutator formula for $u \in L_{\mathfrak{g}}(k,\lambda_2)(0)$ is a direct result of the Jacobi identity:

$$[X(m), \mathcal{Y}(u,z)] = z^m \mathcal{Y}(X(0)u, z). \tag{2.2}$$

From now on, we restrict our discussion on affine vertex operator algebras associated to a finite dimensional simple Lie algebra $\mathfrak{g}$ of level k. By abusing of notation, we use λ to denote the irreducible $L_{\mathfrak{g}}(k,0)$-module $L_{\mathfrak{g}}(k,\lambda)$ and λ' to denote the contragredient module of λ.

Let $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ be four irreducible $L_{\mathfrak{g}}(k,0)$-modules, and fix a basis of intertwining operators as in §2.4. It is proved [KZ, KT] that

$$span\{E\langle u_{\lambda_4'}, \mathcal{Y}^{\lambda_4}_{\lambda_3,\mu;i}(u_{\lambda_3}, z_1)\mathcal{Y}^{\mu}_{\lambda_2,\lambda_1;j}(u_{\lambda_2}, z_2)u_{\lambda_1}\rangle | i,j,\mu\} \tag{2.3}$$

$$= span\{E\langle u_{\lambda_4'}, \mathcal{Y}^{\lambda_4}_{\lambda_2,\gamma;k}(u_{\lambda_2}, z_2)\mathcal{Y}^{\gamma}_{\lambda_3,\lambda_1;l}(u_{\lambda_3}, z_1)u_{\lambda_1}\rangle | k,l,\gamma\}, \tag{2.4}$$

where $u_{\lambda_i} \in L_{\mathfrak{g}}(k,\lambda_i)$. Then there exist $(B^{\lambda_3,\lambda_2}_{\lambda_4,\lambda_1})^{i,j;k,l}_{\mu,\gamma} \in \mathbb{C}$ such that

$$\begin{aligned} &E\langle u_{\lambda_4'}, \mathcal{Y}^{\lambda_4}_{\lambda_3,\mu}(u_{\lambda_3}, z_1)\mathcal{Y}^{\mu}_{\lambda_2,\lambda_1}(u_{\lambda_2}, z_2)u_{\lambda_1}\rangle \\ &\quad = \sum_{k,l,\gamma} (B^{\lambda_3,\lambda_2}_{\lambda_4,\lambda_1})^{i,j;k,l}_{\mu,\gamma} E\langle u_{\lambda_4'}, \mathcal{Y}^{\lambda_4}_{\lambda_2,\gamma;k}(u_{\lambda_2}, z_2)\mathcal{Y}^{\gamma}_{\lambda_3,\lambda_1;l}(u_{\lambda_3}, z_1)u_{\lambda_1}\rangle, \end{aligned} \tag{2.5}$$

(see [H1, H2]). $B^{\lambda_3,\lambda_2}_{\lambda_4,\lambda_1}$ is called the braiding matrix. In the $\mathfrak{sl}(2)$ case, since the fusion rule is either 0 or 1, the braiding matrix can be simply denoted by $(B^{\lambda_3,\lambda_2}_{\lambda_4,\lambda_1})_{\mu,\gamma}$, since $i,j,k,l \in \{0,1\}$.

Now let us turn our discussion to KZ equations for $L_{\mathfrak{g}}(k,0)$. For any intertwining operators $\mathcal{Y}_2 \in \mathcal{V}^{\lambda_4}_{\lambda_3,\lambda}$ and $\mathcal{Y}_1 \in \mathcal{V}^{\lambda}_{\lambda_2,\lambda_1}$, and $u_i \in L_{\lambda_i} = L_{\mathfrak{g}}(k,\lambda_i)(0)$, $i = 1,2,3$, $u_4' \in L_{\lambda_4'} = L_{\mathfrak{g}}(k,\lambda_4)(0)^*$, the function $\Psi(\mathcal{Y}_2, \mathcal{Y}_1, z_1, z_2)$ is defined by:

$$\Psi(\mathcal{Y}_2, \mathcal{Y}_1, z_1, z_2)(u_4 \otimes u_3 \otimes u_2 \otimes u_1) = \langle u_4', \mathcal{Y}_2(u_3, z_1)\mathcal{Y}_1(u_2, z_2)u_1\rangle.$$

Lemma 2.6. *In the region $|z_1| > |z_2| > 0$, this function has a convergent Lauren expansion ([KT]):*

$$\Psi(\mathcal{Y}_2, \mathcal{Y}_1, z_1, z_2)(u_4 \otimes u_3 \otimes u_2 \otimes u_1) = z_2^{\Delta(\mathcal{Y}_1)+\Delta(\mathcal{Y}_2)} \sum_{n \geq 0} \langle u_4', (u_3)_{n-1}(u_2)_{-n-1}u_1 (\frac{z_2}{z_1})^{n-\Delta(\mathcal{Y}_2)}\rangle.$$

Proof. Direct calculation gives

$$\begin{aligned} &\Psi(\mathcal{Y}_2, \mathcal{Y}_1, z_1, z_2)(u_4 \otimes u_3 \otimes u_2 \otimes u_1) \\ =&\langle u_4', \mathcal{Y}_2(u_3, z_1)\mathcal{Y}_1(u_2, z_2)u_1\rangle \\ =&\langle u_4', \sum_{n,m \in \mathbb{Z}} (u_3)_n (u_2)_m u_1 z_1^{-n-1+\Delta(\mathcal{Y}_2)} z_2^{-m-1+\Delta(\mathcal{Y}_1)}\rangle \\ =&\langle u_4', \sum_{n \geq 0} (u_3)_{n-1}(u_2)_{-n-1}u_1 z_1^{-n+\Delta(\mathcal{Y}_2)} z_2^{n+\Delta(\mathcal{Y}_1)}\rangle \\ =&z_1^{\Delta(\mathcal{Y}_2)} z_2^{\Delta(\mathcal{Y}_1)} \sum_{n \geq 0} \langle u_4', (u_3)_{n-1}(u_2)_{-n-1}u_1 (\frac{z_2}{z_1})^n\rangle \\ =&z_2^{\Delta(\mathcal{Y}_1)+\Delta(\mathcal{Y}_2)} \sum_{n \geq 0} \langle u_4', (u_3)_{n-1}(u_2)_{-n-1}u_1 (\frac{z_2}{z_1})^{n-\Delta(\mathcal{Y}_2)}\rangle. \end{aligned}$$

□

Introduce a variable $\xi = \frac{z_2}{z_1}$, then the function $z_2^{-\Delta(\mathcal{Y}_1)-\Delta(\mathcal{Y}_2)}\Psi(\mathcal{Y}_2, \mathcal{Y}_1, z_1, \xi z_1)$ is independent of z_1. We abbreviate it to $\Psi(\mathcal{Y}_2, \mathcal{Y}_1, \xi)$.

In the case that $u_1 \in L_{\lambda_1} = L_{\mathfrak{g}}(k, \lambda_1)(0)$ is the highest weight vector of $\mathfrak{g}$ and $u_4' \in L_{\mathfrak{g}}(k, \lambda_4)(0)^*$ the lowest weight vector of $\mathfrak{g}$, $\Psi(\mathcal{Y}_2, \mathcal{Y}_1, \xi)(u_4', u_3, u_2, u_1)$ verifies the reduced KZ equation [KT]:

$$(k + h^\vee)\frac{d}{d\xi}\Psi = \frac{\Omega_{1,2} - (k + h^\vee)(\Delta(\mathcal{Y}_1) + \Delta(\mathcal{Y}_2))}{\xi}\Psi + \frac{\Omega_{2,3}}{\xi - 1}\Psi, \tag{2.6}$$

where Ω is the Casimir element

$$\Omega = \sum_{i=1}^{d} u^{(i)} \otimes u^{(i)},$$

and

$$\Omega_{1,2}\Psi(\mathcal{Y}_2, \mathcal{Y}_1, \xi)(u_4', u_3, u_2, u_1) = \sum_{i=1}^{d} \Psi(\mathcal{Y}_2, \mathcal{Y}_1, \xi)(u_4', u_3, a^{(i)}u_2, a^{(i)}u_1),$$

$$\Omega_{2,3}\Psi(\mathcal{Y}_2, \mathcal{Y}_1, \xi)(u_4', u_3, u_2, u_1) = \sum_{i=1}^{d} \Psi(\mathcal{Y}_2, \mathcal{Y}_1, \xi)(u_4', a^{(i)}u_3, a^{(i)}u_2, u_1).$$

The solutions of the reduced KZ equations can only have poles of finite order.

2.6. **Crossing symmetry of four point functions.** Crossing symmetry is an important property of quantum groups. Much study was devoted to the connection between conformal field theory and representations of the braid group ([KT, V], et). It is expected that the crossing symmetry of the correlation functions in conformal field theory comes from the crossing symmetry of the quantum group relation. The cases of minimal series were carefully treated in [FFK]. For the WZW $SU(2)$ model, the elements of braiding matrices of the correlation functions are essentially the quantum $6j$ symbols [HSWY]. Following from [AGS, DF1, DF2, HSWY], the crossing symmetry of the braiding matrix of the correlation functions in the WZW $SU(2)$ model is derived, and is related to the symmetry of the quantum $6j$ symbols.

Now fix a basis of intertwining operators $\mathcal{Y}_{i,j}^k \in \mathcal{V}_{i,j}^k$ in the $L_{\mathfrak{sl}(2)}(k, 0)$ case (here $N_{i,j}^k$ is either 0 or 1). We use the notation from §2.4 and express the braiding matrix in the following way:

$$E\langle u_k', \mathcal{Y}_{j,a}^k(u_j, w)\mathcal{Y}_{i,l}^a(u_i, z)u_l\rangle = \sum_b (B_{k,l}^{j,i})_{a,b} E\langle u_k', \mathcal{Y}_{i,b}^k(u_i, z)\mathcal{Y}_{j,l}^b(u_j, w)u_l\rangle.$$

The crossing symmetry can be explained as the following lemma (cf. Page 657 of [FFK]):

Lemma 2.7. *The braiding matrix under the basis of intertwining operators chosen above satisfies*

$$(B_{k,l}^{j,i})_{a,b} T_{j,a}^k T_{i,l}^a = (B_{k,l}^{i,j})_{b,a} T_{i,b}^k T_{j,l}^b,$$

where $T_{m,n}^r \in \mathbb{C}$ is a constant uniquely determined by the vertex of type $\begin{pmatrix} & r & \\ m & & n \end{pmatrix}$.

2.7. **Level-Rank Duality.** Level-rank duality has been explained by different methods in [GW, SA, NT]. We will be interested in the following conformal inclusion:

$$L_{\mathfrak{sl}(m)}(n, 0) \otimes L_{\mathfrak{sl}(n)}(m, 0) \subset L_{\mathfrak{sl}(mn)}(1, 0). \tag{2.7}$$

In the classification of conformal inclusions in [GNO], the above conformal inclusion corresponds to AIII.

The decomposition of $L_{\mathfrak{sl}(mn)}(1, 0)$ under $L_{\mathfrak{sl}(m)}(n, 0) \otimes L_{\mathfrak{sl}(n)}(m, 0)$ is known (see [ABI, X1]). To describe such a decomposition, let us prepare some notations. The level n (resp. m) dominant integral weight of $\hat{\mathfrak{sl}}(m)$ (resp. $\hat{\mathfrak{sl}}(n)$) will be denoted by λ (resp. $\dot{\lambda}$). P_{++}^n (resp. $\dot{P}_{++}^m$) denote the set of highest weights of level n of $\hat{\mathfrak{sl}}(m)$ (resp. level m of $\hat{\mathfrak{sl}}(n)$). The fundamental weight of $\mathfrak{sl}(m)$ (resp. $\mathfrak{sl}(n)$) will be denoted by Λ_i (resp. $\dot{\Lambda}_j$). We will use Λ_0 (resp. $\dot{\Lambda}_0$) or 0 (resp.

$\dot{0}$) to denote the trivial representation of $\mathfrak{sl}(m)$ (resp. $\mathfrak{sl}(n)$). Then any λ can be expressed as $\lambda = \sum_{i=0}^{m-1} \lambda_i \Lambda_i$, and $\sum_{i=0}^{m-1} \lambda_i = n$. Instead of $\lambda = (\lambda_0, \cdots, \lambda_{m-1})$, it will be more convenient to use

$$\lambda + \rho = \sum_{i=0}^{m-1} \lambda'_i \Lambda_i$$

with $\lambda'_i = \lambda_i + 1$. Then $\sum_{i=0}^{m-1} \lambda'_i = m + n$.

Due to the cyclic symmetry of the extended Dynkin diagram of $\mathfrak{sl}(m)$, the group $\mathbb{Z}_m$ acts on P_{++}^n by

$$\Lambda_i \to \Lambda_{(i+\mu) mod m}, \ \mu \in \mathbb{Z}_m.$$

Let $\Omega_{m,n} = P_{++}^n / \mathbb{Z}_m$. Then there is a natural bijection between $\Omega_{m,n}$ and $\Omega_{n,m}$ (see §2 of [ABI]).

We parameterize the bijection by a map

$$\beta : P_{++}^n \to \dot{P}_{++}^m$$

as follows. Set

$$r_j = \sum_{i=j}^{m} \lambda'_i, \ 1 \le j \le m,$$

where $\lambda'_m \equiv \lambda'_0$. The sequence $(r_1, \cdots, r_m)$ is decreasing, $m + n = r_1 > r_2 > \cdots > r_m \ge 1$. Take the complementary sequence $(\bar{r}_1, \bar{r}_2, \cdots, \bar{r}_n)$ in $\{1, 2, \cdots, m+n\}$ with $\bar{r}_1 > \bar{r}_2 > \cdots > \bar{r}_n$. Put

$$s_j = m + n + \bar{r}_n - \bar{r}_{n-j+1}, \ 1 \le j \le n.$$

Then $m + n = s_1 > s_2 > \cdots > s_n \ge 1$. The map β is defined by

$$(r_1, \cdots, r_m) \to (s_1, \cdots, s_n).$$

The following lemmas summarizes what we will use.

Lemma 2.8. [X1] *Let Q be the root lattice of $\mathfrak{sl}(m)$, Λ_i, $0 \le i \le m-1$, its fundamental weights, and $Q_i = (Q + \Lambda_i) \cap P_{++}^n$. Let $\tilde{\Lambda} \in \mathbb{Z}_{mn}$ denote a level 1 highest weight of $\mathfrak{sl}(mn)$ and $\lambda \in Q_{\tilde{\Lambda} mod m}$. Then there exists a unique $\dot{\lambda} \in \dot{P}_{++}^m$ with $\dot{\lambda} = \mu\beta(\lambda)$ for some unique $\mu \in \mathbb{Z}_n$ such that $L_{\mathfrak{sl}(m)}(n, \lambda) \otimes L_{\mathfrak{sl}(n)}(m, \dot{\lambda})$ appears once and only once in $L_{\mathfrak{sl}(mn)}(1, \tilde{\Lambda})$. The map*

$$\lambda \to \dot{\lambda} = \mu\beta(\lambda)$$

is one-to-one. Moreover, $L_{\mathfrak{sl}(mn)}(1, \tilde{\Lambda})$ is a direct sum of all $L_{\mathfrak{sl}(m)}(n, \lambda) \otimes L_{\mathfrak{sl}(n)}(m, \dot{\lambda})$:

$$L_{\mathfrak{sl}(mn)}(1, \tilde{\Lambda}) = \bigoplus_{\lambda \in Q_{\tilde{\Lambda} mod m}} L_{\mathfrak{sl}(m)}(n, \lambda) \otimes L_{\mathfrak{sl}(n)}(m, \dot{\lambda}).$$

Lemma 2.9. *Take $\tilde{\Lambda} = 0$ in the above lemma, $\{\lambda | \lambda \in Q_0\}$ (resp. $\{\dot{\lambda}\}$) is closed under fusion. Moreover, the map $\lambda \to \dot{\lambda}$ gives an isomorphism between the two fusion subalgebras.*

Remark 2.10. *In the case $m = 2$, $n = 10$ and $\tilde{\Lambda} = 0$, $Q_0 = \{0, 2, 4, 6, 8, 10\}$. When we take $m = 2$, $n = 28$, $Q_0 = \{0, 2, 4, \cdots, 28\}$.*

We write the conformal net and subnet which correspond to $L_{\mathfrak{sl}(m)}(n, 0) \otimes L_{\mathfrak{sl}(n)}(m, 0) \subset L_{\mathfrak{sl}(mn)}(1, 0)$ as $\mathcal{A} \subset \mathcal{B}$. For simplicity we assume that the spectrum of $\mathcal{A} \subset \mathcal{B}$ is $\sum_\lambda \lambda \otimes (1, \lambda)$ where λ, $(1, \lambda)$ label the irreducible representations of $L_{\mathfrak{sl}(m)}(n, 0)$ and $L_{\mathfrak{sl}(n)}(m, 0)$ respectively.

Lemma 2.11. *For $U_{\bar{\lambda}} \in Hom(\alpha_{\bar{\lambda}}, \alpha_{(1,\lambda)})$ be a unitary as in (3) of Prop. 3.7 in* [X2]*, $T_\lambda \in Hom(1, \alpha_{\lambda,(1,\lambda)})$, one has $U_{\bar{\lambda}} = n_\lambda T_\lambda$, for some $n_\lambda \in \mathcal{A}(I)$.*

Proof. Let $r_\lambda \neq 0$, $r_\lambda \in \mathrm{Hom}(\lambda\bar{\lambda}, 1)$. Then $(1, \lambda)(r_\lambda) T_\lambda \in \mathrm{Hom}(\alpha_{\bar{\lambda}}, \alpha_{(1,\lambda)})$, and $(1, \lambda)(r_\lambda) T_\lambda \neq 0$. It follows that $U_{\bar{\lambda}} = n_\lambda T_\lambda$, for some $n_\lambda \in \mathcal{A}(I)$. □

Proposition 2.12. *If $\lambda_3 \prec \lambda_1\lambda_2$, then $E(T_{\lambda_3}T^*_{\lambda_1}T^*_{\lambda_2}) \neq 0$, where $T_{\lambda_i} \in Hom(1, \alpha_{\lambda_i}), i = 1, 2, 3$.*

Proof. Choose $T \neq 0$, $T \in \mathrm{Hom}(\overline{\lambda_3}, \overline{\lambda_1\lambda_2})$. Then $U_{\lambda_2}U_{\lambda_1}TU^*_{\lambda_3} \in \mathrm{Hom}(\alpha_{(1,\lambda_3)}, \alpha_{(1,\lambda_1)(\alpha,\lambda_2)}) = \mathrm{Hom}((1,\lambda_3),(1,\lambda_1)(1,\lambda_2)) \subset \mathcal{A}(I)$. So $E(U_{\lambda_2}U_{\lambda_1}TU^*_{\lambda_3}) = U_{\lambda_2}U_{\lambda_1}TU^*_{\lambda_3} \neq 0$.

By the Lemma 2.11,

$$\begin{aligned} E(U_{\lambda_2}U_{\lambda_1}TU^*_{\lambda_3}) \quad &= E(n_{\lambda_2}T_{\lambda_2}n_{\lambda_1}T_{\lambda_1}TT^*_{\lambda_3}n^*_{\lambda_3}) \\ &= E(n_{\lambda_2}\lambda_2(n_{\lambda_1})T_{\lambda_2}T_{\lambda_1}T^*_{\lambda_3}\lambda_3(T)n^*_{\lambda_3}) \\ &= n_{\lambda_2}\lambda_2(n_{\lambda_1})E(T_{\lambda_2}T_{\lambda_1}T^*_{\lambda_3})\lambda_3(T)n^*_{\lambda_3}) \\ &\neq 0. \end{aligned}$$

It follows that $E(T_{\lambda_2}T_{\lambda_1}T^*_{\lambda_3}) \neq 0$. Taking the adjoint we have proved the Proposition. $\square$

Corollary 2.13. *For inclusions $SU(n)_m \times SU(m)_n \subset SU(mn)_1$, suppose the vertex operator*

$$J^{(\lambda_2,\dot{\lambda_2})}(z) = \sum_{\lambda_1,\lambda_3} D^{\lambda_3}_{\lambda_2,\lambda_1}\mathcal{Y}^{\lambda_3}_{\lambda_2,\lambda_1}(\cdot,z) \otimes \mathcal{Y}^{\dot{\lambda}_3}_{\dot{\lambda}_2,\dot{\lambda}_1}(\cdot,z).$$

If $\lambda_3 \prec \lambda_2\lambda_1$, then $D^{\lambda_3}_{\lambda_2,\lambda_1} \neq 0$.

Proof. If $D^{\lambda_3}_{\lambda_2,\lambda_1} = 0$, then $H_{\lambda_3} \otimes H_{\dot{\lambda}_3} \perp J^{(\lambda_2,\dot{\lambda_2})}J^{(\lambda_1,\dot{\lambda}_1)}H_0 \otimes H_{\dot{0}}$. By Proposition 2.12 and (2) of Lemma 3.3 in [X4], we have $H_{\lambda_3} \otimes H_{\dot{\lambda}_3} \subset J^{(\lambda_2,\dot{\lambda}_2)}J^{(\lambda_1,\dot{\lambda}_1)}H_0 \otimes H_{\dot{0}}$, where $J^{(\lambda_2,\dot{\lambda}_2)} = V(\lambda_2,\dot{\lambda}_2)$ in (2) of Lemma 3.3 in [X4], a contradiction. $\square$

Remark 2.14. *A proof of Corollary 2.13 using vertex operator algebra language has not been found in this paper. We will do direct calculation in the case we need for constructing the main examples (see Remark 3.6 and §7).*

2.8. **Lattice Vertex Operator Algebras.** Let L be a rank d even lattice with a positive definite symmetric $\mathbb{Z}$-bilinear form $(\cdot,\cdot)$. We set $\mathfrak{h} = \mathbb{C} \otimes_{\mathbb{Z}} L$ and extend $(\cdot,\cdot)$ to a $\mathbb{C}$-bilinear form on $\mathfrak{h}$. Let $\hat{\mathfrak{h}} = \mathbb{C}[t,t^{-1}] \otimes \mathfrak{h} \oplus \mathbb{C}C$ be the affinization of commutative Lie algebra $\mathfrak{h}$. For any $\lambda \in \mathfrak{h}$, we can define a one dimensional $\hat{\mathfrak{h}}^+$-module $\mathbb{C}e^\lambda$ by the actions $\rho(h \otimes t^m)e^\lambda = (\lambda,h)\delta_{m,0}e^\lambda$ and $\rho(C)e^\lambda = e^\lambda$ for $h \in \mathfrak{h}$ and $m \geq 0$. Now we denote by

$$M(1,\lambda) = U(\hat{\mathfrak{h}}) \otimes_{U(\hat{\mathfrak{h}}^+)} \mathbb{C}e^\lambda \cong S(t^{-1}\mathbb{C}[t^{-1}])$$

the $\hat{\mathfrak{h}}$-module induced from $\hat{\mathfrak{h}}^+$-module. Set $M(1) = M(1,0)$. Then there exists a linear map $Y : M(1) \to (\mathrm{End}M(1,\lambda))[[z,z^{-1}]]$ such that $(M(1),Y,\mathbf{1},\omega)$ is a simple vertex operator algebra and $(M(1,\lambda),Y)$ becomes an irreducible $M(1)$-module for any $\lambda \in \mathfrak{h}$ (see [FLM]). The vacuum vector and the Virasoro element are given by $\mathbf{1} = e^0$ and $\omega = \frac{1}{2}\sum_{i=1}^d a_i(-1)^2 \otimes e^0$, respectively, where $\{a_i\}$ is an orthonormal basis of $\mathfrak{h}$.

Let L be any positive definite even lattice and let $\hat{L}$ be the canonical central extension of L by the cyclic group $\langle\kappa\rangle$ of order 2:

$$1 \to \langle\kappa\rangle \to \hat{L} \bar{\to} L \to 0$$

with the commutator map $c(\alpha,\beta) = \kappa^{(\alpha,\beta)}$ for $\alpha,\beta \in L$. Let $e : L \to \hat{L}$ be a section such that $e_0 = 1$ and $\varepsilon : L \times L \to \langle\kappa\rangle$ be the corresponding 2-cocycle. We can assume that ε is bimultiplicative. Then $\varepsilon(\alpha,\beta)\varepsilon(\beta,\alpha) = \kappa^{(\alpha,\beta)}$,

$$\varepsilon(\alpha,\beta)\varepsilon(\alpha+\beta,\gamma) = \varepsilon(\beta,\gamma)\varepsilon(\alpha,\beta+\gamma)$$

and $e_\alpha e_\beta = \varepsilon(\alpha,\beta)e_{\alpha+\beta}$ for $\alpha,\beta,\gamma \in L$.

Let $L^\circ = \{\lambda \in \mathfrak{h} | (\alpha,\lambda) \in \mathbb{Z}\}$ be the dual lattice of L. Then there is a $\hat{L}$-module structure on $\mathbb{C}[L^\circ] = \bigoplus_{\lambda \in L^\circ} \mathbb{C}e^\lambda$ such that κ acts as -1 (see [DL]). Let $L^\circ = \bigcup_{i \in L^\circ/L}(L+\lambda_i)$ be the coset

decomposition such that $\lambda_0 = 0$. Set $\mathbb{C}[L+\lambda_i] = \bigoplus_{\alpha\in L} \mathbb{C}e^{\alpha+\lambda_i}$. Then $\mathbb{C}[L^\circ] = \bigoplus_{i\in L^\circ/L} \mathbb{C}[L+\lambda_i]$ and each $\mathbb{C}[L+\lambda_i]$ is an $\hat{L}$-submodule of $\mathbb{C}[L^\circ]$. The action of $\hat{L}$ on $\mathbb{C}[L+\lambda_i]$ is as follows:

$$e_\alpha e^{\beta+\lambda_i} = \varepsilon(\alpha,\beta)e^{\alpha+\beta+\lambda_i}$$

for $\alpha, \beta \in L$.

We can identify e^α with e_α for $\alpha \in L$. For any $\lambda \in L^\circ$, set $\mathbb{C}[L+\lambda] = \bigoplus_{\alpha\in L} \mathbb{C}e^{\alpha+\lambda}$ and define $V_{L+\lambda} = M(1) \otimes \mathbb{C}[L+\lambda]$. Then for any $V_{L+\lambda}$, there exists a linear map

$$Y : V_L \to (\mathrm{End} V_{L_\lambda})[[z, z^{-1}]]$$

such that (V_L, Y, α, ω) becomes a simple vertex operator algebra and $(V_{L+\lambda}, Y)$ is an irreducible V_L-module [B, FLM]. And $V_{L+\lambda_i}$ for $\lambda_i \in L^\circ/L$ give all inequivalent irreducible V_L-modules (see [D]). The vertex operator $Y(h(-1)\mathbf{1}, z)$ and $Y(e^\alpha, z)$ associated to $h(-1)\mathbf{1}$ and e^α are defined as

$$Y(h(-1)\mathbf{1}, z) = h(z) = \sum_{n\in\mathbb{Z}} h(-n)z^{-n-1},$$

$$Y(e^\alpha, z) = \exp(\sum_{n=1}^{\infty} \frac{\alpha(-n)}{n} z^n) \exp(-\sum_{n=1}^{\infty} \frac{\alpha(n)}{n} z^{-n}) e_\alpha z^\alpha,$$

where $h(-n)$ is the action of $h \otimes t^n$ on $V_{L+\lambda}$, e_α is the left action of $\hat{L}$ on $\mathbb{C}[L^\circ]$, and z^α is the operator on $\mathbb{C}[L^\circ]$ defined by $z^\alpha e^\lambda = z^{(\alpha,\lambda)} e^\lambda$. The vertex operator associated to the vector $v = \beta_1(-n_1)\cdots\beta_r(-n_r)e^\alpha$ for $\beta_i \in \mathfrak{h}$, $n_i \geq 1$, and $\alpha \in L$ is defined as

$$Y(v, z) =: \partial^{(n_1-1)}\beta_1(z)\cdots\partial^{(n_r-1)}\beta_r(z)Y(e^\alpha, z) :,$$

where $\partial = (1/n!)(d/dz)$ and $:,:$ is the normal ordered products.

In the case we choose L be the root lattice of simple Lie algebras $\mathfrak{g}$ of ADE type, one knows $V_L \cong L_{\mathfrak{g}}(1, 0)$ as vertex operator algebras.

3. The mirror extension of $L_{\mathfrak{sl}(10)}(2, 0)$

This section is devoted to the construction of vertex operator algebra

$$V = L_{\mathfrak{sl}(10)}(2, 0)^e = L_{\mathfrak{sl}(10)}(2, 0) \oplus L_{\mathfrak{sl}(10)}(2, \Lambda_3 + \Lambda_7)$$

based on the conformal inclusions

$$SU(2)_{10} \subset Spin(5)_1 \quad (L_{\mathfrak{sl}(2)}(10, 0) \subset L_{B_2}(1, 0)) \tag{3.1}$$

and

$$SU(2)_{10} \times SU(10)_2 \subset SU(20)_1 \quad (L_{\mathfrak{sl}(2)}(10, 0) \otimes L_{\mathfrak{sl}(10)}(2, 0) \subset L_{\mathfrak{sl}(20)}(1, 0)). \tag{3.2}$$

3.1. **The conformal inclusions.**

The conformal inclusion (3.1) is well studied in conformal nets theory. Due to [CIZ] and a recent result of [DLN], the corresponding conformal inclusion of vertex operator algebras and the branching rules are also established in vertex operator algebra theory. The decomposition of $L_{B_2}(1, 0)$ as an $L_{\mathfrak{sl}(2)}(10, 0)$-module is as follow:

$$L_{B_2}(1, 0) = L_{\mathfrak{sl}(2)}(10, 0) \oplus L_{\mathfrak{sl}(2)}(10, 6).$$

For convenience, we denote the above decomposition as

$$L_{B_2}(1, 0) = 0 + 6. \tag{3.3}$$

The conformal inclusion (3.2) comes from the level-rank duality (§2.7) and the branching rules are given in Lemma 2.8 and Remark 2.10:

$$
\begin{aligned}
L_{\mathfrak{sl}(20)}(1,0)) = & L_{\mathfrak{sl}(2)}(10,0) \otimes L_{\mathfrak{sl}(10)}(2,0) \\
& \oplus L_{\mathfrak{sl}(2)}(10,2) \otimes L_{\mathfrak{sl}(10)}(2,\Lambda_1+\Lambda_9) \\
& \oplus L_{\mathfrak{sl}(2)}(10,4) \otimes L_{\mathfrak{sl}(10)}(2,\Lambda_2+\Lambda_8) \\
& \oplus L_{\mathfrak{sl}(2)}(10,6) \otimes L_{\mathfrak{sl}(10)}(2,\Lambda_3+\Lambda_7) \\
& \oplus L_{\mathfrak{sl}(2)}(10,8) \otimes L_{\mathfrak{sl}(10)}(2,\Lambda_4+\Lambda_6) \\
& \oplus L_{\mathfrak{sl}(2)}(10,10) \otimes L_{\mathfrak{sl}(10)}(2,2\Lambda_5)
\end{aligned} \tag{3.4}
$$

We will use the notation as in Lemma 2.8 for the above decomposition:

$$L_{\mathfrak{sl}(20)}(1,0)) = \sum_{\lambda \text{ even},\lambda=0}^{10} \lambda \times \dot{\lambda}.$$

The decompositions (3.3) and (3.4) and the Mirror Extension Conjecture allow us to make the following assertion, which is the first main theorem of this paper:

Theorem 3.1. *There is a vertex operator algebra structure on*

$$V = L_{\mathfrak{sl}(10)}(2,0)^e = \dot{0} + \dot{6} = L_{\mathfrak{sl}(10)}(2,0) \oplus L_{\mathfrak{sl}(10)}(2,\Lambda_3+\Lambda_7).$$

3.2. **The construction of the VOA extension.** In this section, we focus on defining the vertex operator

$$\dot{Y}(\cdot,z) : V \to \mathrm{End}V[[z,z^{-1}]]$$

that gives a vertex operator algebra structure on $L_{\mathfrak{sl}(10)}(2,0)^e$.

First we use $Y(\cdot,z)$ and $\tilde{Y}(\cdot,z)$ to denote the vertex operators of $L_{B_2}(1,0)$ and $L_{\mathfrak{sl}(20)}(1,0)$ respectively. We fix a basis of the intertwining operators $\mathcal{Y}_{a,b}^{c}$ (resp. $\mathcal{Y}_{\dot{a},\dot{b}}^{\dot{c}}$) among modules $\{\lambda \in Q_0\}$ (resp. $\{\dot{\lambda}\}$) of $L_{\mathfrak{sl}(2)}(10,0)$ (resp. $L_{\mathfrak{sl}(10)}(2,0)$) such that

$$\tilde{Y}(u_1 \otimes u_2, z) = \sum_{\lambda_1,\lambda_2 \in Q_0} D_{\lambda,\lambda_1}^{\lambda_2} \mathcal{Y}_{\lambda,\lambda_1}^{\lambda_2}(u_1,z) \otimes \mathcal{Y}_{\dot{\lambda},\dot{\lambda}_1}^{\dot{\lambda}_2}(u_2,z), \tag{3.5}$$

for $u_1 \in L_{\mathfrak{sl}(2)}(10,\lambda)$, $u_2 \in L_{\mathfrak{sl}(10)}(2,\dot{\lambda})$, $\mathcal{Y}_{\lambda,\lambda_1}^{\lambda_2} \in \mathcal{V}_{\lambda,\lambda_1}^{\lambda_2}$. One can choose $D_{\lambda,\lambda_1}^{\lambda_2} = \delta_{1,N_{\lambda,\lambda_1}^{\lambda_2}}$ because of Corollary 2.13 by suitably choosing the basis of intertwining operators $\mathcal{Y}_{a,b}^{c}$ and $\mathcal{Y}_{\dot{a},\dot{b}}^{\dot{c}}$. But we want a pure vertex operator algebra proof that $D_{\lambda,\lambda_1}^{\lambda_2} \neq 0$ if $N_{\lambda,\lambda_1}^{\lambda_2} \neq 0$. It turns out that we only need to show that $D_{6,6}^{\mu} \neq 0$, $D_{6,\mu}^{6} \neq 0$ for $\mu = 0,2,4,6,8$ for the purpose of this paper. This result is given in §7. So in the discussion below we always assume that $D_{6,6}^{\mu} = D_{6,\mu}^{6} = D_{\lambda,\lambda_1}^{\lambda_2} = 1$ with $N_{\lambda,\lambda_1}^{\lambda_2} \neq 0$ for $\mu = 0,2,4,6,8$ and one of $\lambda,\lambda_1,\lambda_2$ being 0.

Under the same basis of intertwining operators, for $u \in L_{\mathfrak{sl}(2)}(10,0) \subset L_{\mathfrak{sl}(2)}(10,0)^e$ the vertex operator is obviously of the form

$$Y(u,z) = J^0(u,z) = Y_0(u,z) + Y_6(u,z).$$

For $u \in L_{\mathfrak{sl}(2)}(10,6)$, we denote $Y(u,z)$ by

$$J^6(u,z) = Y(u,z) = \sum_{\lambda_1,\lambda_2 \in \{0,6\}} c_{6,\lambda_1}^{\lambda_2} \mathcal{Y}_{6,\lambda_1}^{\lambda_2}(u,z),$$

where $c_{6,\lambda_1}^{\lambda_2} \in \mathbb{C}$. Similarly, we write

$$\dot{J}^{\dot{0}}(u,z) = \dot{Y}(u,z) = Y_{\dot{0}}(u,z) + Y_{\dot{6}}(u,z)$$

for $u \in L_{\mathfrak{sl}(10)}(2, \dot{0})$ and

$$\dot{J}^{\dot{6}}(u,z) = \dot{Y}(u,z) = \sum_{\lambda_1,\lambda_2\in\{0,6\}} c_{\dot{6},\dot{\lambda}_1}^{\dot{\lambda}_2} \mathcal{Y}_{\dot{6},\dot{\lambda}_1}^{\dot{\lambda}_2}(u,z)$$

for $u \in L_{\mathfrak{sl}(10)}(2, \Lambda_3 + \Lambda_7)$, where $c_{\dot{6},\dot{\lambda}_1}^{\dot{\lambda}_2} \in \mathbb{C}$ are the coefficients needed to be determined such that $\dot{Y}$ satisfying locality.

Notice that $L_{B_2}(1,0)$ is self dual. Then for any $u_i \in L_{B_2}(1,0)$, $i = 1,2,3,4$,

$$E\langle u_4, Y(u_3,w)Y(u_2,z)u_1\rangle$$

is a rational symmetric function since $L_{B_2}(1,0)$ is a vertex operator algebra. If we choose

$$u_i \in L_{\mathfrak{sl}(2)}(10,\lambda_i) \subset L_{B_2}(1,0),\ i = 1,2,3,4,$$

where $\lambda_i \in \{0,6\}$, then we have

$$\begin{aligned}
&E\langle u_4, Y(u_3,w)Y(u_2,z)u_1\rangle\\
=&E\langle u_4, J^{\lambda_3}(u_3,w)J^{\lambda_2}(u_2,z)u_1\rangle\\
=&\sum_{\mu\in\{0,6\}} c_{\lambda_3,\mu}^{\lambda_4} c_{\lambda_2,\lambda_1}^{\mu} E\langle u_4, \mathcal{Y}_{\lambda_3,\mu}^{\lambda_4}(u_3,w)\mathcal{Y}_{\lambda_2,\lambda_1}^{\mu}(u_2,z)u_1\rangle\\
=&\sum_{\mu,\gamma\in\{0,6\}} c_{\lambda_3,\mu}^{\lambda_4} c_{\lambda_2,\lambda_1}^{\mu} (B_{\lambda_4,\lambda_1}^{\lambda_3,\lambda_2})_{\mu,\gamma} E\langle u_4, \mathcal{Y}_{\lambda_2,\gamma}^{\lambda_4}(u_2,z)\mathcal{Y}_{\lambda_3,\lambda_1}^{\gamma}(u_3,w)u_1\rangle\\
=&\sum_{\gamma\in\{0,6\}} c_{\lambda_2,\gamma}^{\lambda_4} c_{\lambda_3,\lambda_1}^{\gamma} E\langle u_4, \mathcal{Y}_{\lambda_2,\gamma}^{\lambda_4}(u_2,z)\mathcal{Y}_{\lambda_3,\lambda_1}^{\gamma}(u_3,w)u_1\rangle\\
=&E\langle u_4, J^{\lambda_2}(u_2,z)J^{\lambda_3}(u_3,w)u_1\rangle\\
=&E\langle u_4, Y(u_2,z)Y(u_3,w)u_1\rangle
\end{aligned} \tag{3.6}$$

by using the braiding isomorphism and locality of correlation functions of $L_{B_2}(1,0)$.

Due to the linearly independent property of the correlation functions (see §2.4), we have

$$\Sigma_{\mu\in\{0,6\}} c_{\lambda_3,\mu}^{\lambda_4} c_{\lambda_2,\lambda_1}^{\mu} (B_{\lambda_4,\lambda_1}^{\lambda_3,\lambda_2})_{\mu,\gamma} = c_{\lambda_2,\gamma}^{\lambda_4} c_{\lambda_3,\lambda_1}^{\gamma} \tag{3.7}$$

Lemma 3.2. *Equation (3.7) is a necessary and sufficient condition for*

$$E\langle u_4, J^{\lambda_3}(u_3,w)J^{\lambda_2}(u_2,z)u_1\rangle$$

to be a symmetric rational function of w, z, for primary vectors $u_i \in L_{\mathfrak{sl}(2)}(10,\lambda_i)(0)$, $i = 2,3$, u_1 the highest weight vector of $\mathfrak{sl}(2)$ in $L_{\mathfrak{sl}(2)}(10,\lambda_1)(0)$ and u_4 the lowest weight vector of $\mathfrak{sl}(2)$ in $L_{\mathfrak{sl}(2)}(10,\lambda_4)(0)^*$, where

$$J^{\lambda_i} = \sum_{\lambda_j,\lambda_k} c_{\lambda_i,\lambda_j}^{\lambda_k} \mathcal{Y}_{\lambda_i,\lambda_j}^{\lambda_k}.$$

Proof: According to equation (3.6), the condition is clearly necessary. Now assume equation (3.7) holds. Then $E\langle u_4, J^{\lambda_3}(u_3,w)J^{\lambda_2}(u_2,z)u_1\rangle$ is obviously symmetric.

Since

$$\langle u_4, J^{\lambda_3}(u_3,w)J^{\lambda_2}(u_2,z)u_1\rangle = z^{h_{\lambda_4}-h_{\lambda_1}-h_{\lambda_2}-h_{\lambda_3}}\Psi(\xi),$$

where $\xi = \frac{z}{w}$. The $\Psi(\xi)$ satisfies the reduced KZ equation (2.6). Thus $\Psi(\xi)$ is analytic except at 0, 1, ∞, and can only have poles of finite order at 0, 1, ∞. It follows that $\Psi(\xi)$ is a rational function of ξ. $\square$

In order to prove Theorem 3.1, we only need to define the vertex operator on V satisfying locality, which is equivalent to that the four point functions are rational symmetric functions. So it suffices to find solutions to the dotted version of equation (3.7).

The following lemma which is essential in our proof of Theorem 3.1 comes from the locality of vertex operators $\tilde{Y}$ (see equation (3.5)) on $L_{\mathfrak{sl}(20)}(1,0)$.

Lemma 3.3. *We have*

$$\sum_{\mu}(B^{\lambda_3,\lambda_2}_{\lambda_4,\lambda_1})_{\mu,\gamma}\cdot(B^{\dot\lambda_3,\dot\lambda_2}_{\dot\lambda_4,\dot\lambda_1})_{\dot\mu,\dot\gamma_1}=\delta_{\gamma,\dot\gamma_1} \tag{3.8}$$

for $\lambda_i\in\{0,6\},\ i=1,2,3,4.$

Proof. Since every $L_{\mathfrak{sl}(2)}(10,\lambda_i)\otimes L_{\mathfrak{sl}(10)}(2,\dot\lambda_i)$ is a self dual $L_{\mathfrak{sl}(2)}(10,0)\otimes L_{\mathfrak{sl}(10)}(2,\dot 0)$-module, for $u_i\otimes\dot u_i\in L_{\mathfrak{sl}(2)}(10,\lambda_i)\otimes L_{\mathfrak{sl}(10)}(2,\dot\lambda_i)$, $i=1,2,3,4$, the four point function

$$\begin{aligned}&E\langle u_4\otimes\dot u_4,\tilde{Y}(u_3\otimes\dot u_3,w)\tilde{Y}(u_2\otimes\dot u_2,z)u_1\otimes\dot u_1\rangle\\ =&\Sigma_\mu E\langle u_4,\mathcal{Y}^{\lambda_4}_{\lambda_3,\mu}(u_3,w)\mathcal{Y}^{\mu}_{\lambda_2,\lambda_1}(u_2,z)u_1\rangle\otimes\langle\dot u_4,\mathcal{Y}^{\dot\lambda_4}_{\dot\lambda_3,\dot\mu}(\dot u_3,w)\mathcal{Y}^{\dot\mu}_{\dot\lambda_2,\dot\lambda_1}(\dot u_2,z)\dot u_1\rangle\end{aligned} \tag{3.9}$$

is a rational symmetric function.

Since switching λ_2,λ_3, and $\dot\lambda_2,\dot\lambda_3$ gives the same analytic continuation, we have

$$\begin{aligned}&\Sigma_{\mu,\gamma,\dot\gamma_1}E\langle u_4\otimes\dot u_4,(B^{\lambda_3,\lambda_2}_{\lambda_4,\lambda_1})_{\mu,\gamma}\cdot(B^{\dot\lambda_3,\dot\lambda_2}_{\dot\lambda_4,\dot\lambda_1})_{\dot\mu,\dot\gamma_1}\\ &\qquad\mathcal{Y}^{\lambda_4}_{\lambda_2,\gamma}(u_2,z)\mathcal{Y}^{\gamma}_{\lambda_2,\lambda_1}(u_3,w)\otimes\mathcal{Y}^{\dot\lambda_4}_{\dot\lambda_2,\dot\gamma_1}(\dot u_2,z)\mathcal{Y}^{\dot\gamma_1}_{\dot\lambda_3,\dot\lambda_1}(\dot u_3,w)u_1\otimes\dot u_1\rangle\\ &=\Sigma_\gamma E\langle u_4\otimes\dot u_4,\mathcal{Y}^{\lambda_4}_{\lambda_2,\gamma}(u_2,z)\mathcal{Y}^{\gamma}_{\lambda_3,\lambda_1}(u_3,w)\otimes\mathcal{Y}^{\dot\lambda_4}_{\dot\lambda_2,\dot\gamma}(\dot u_2,z)\mathcal{Y}^{\dot\gamma}_{\dot\lambda_3,\dot\lambda_1}(\dot u_3,w)u_1\otimes\dot u_1\rangle.\end{aligned} \tag{3.10}$$

Due to the linear independence of the four point functions for $L_{\mathfrak{sl}(2)}(10,0)\otimes L_{\mathfrak{sl}(10)}(2,0)$ in the above equation, one must have

$$\Sigma_\mu(B^{\lambda_3,\lambda_2}_{\lambda_4,\lambda_1})_{\mu,\gamma}\cdot(B^{\dot\lambda_3,\dot\lambda_2}_{\dot\lambda_4,\dot\lambda_1})_{\dot\mu,\dot\gamma_1}=\delta_{\gamma,\dot\gamma_1} \tag{3.11}$$

as desired. □

We are now in a position to determine the vertex operator on $V=L_{\mathfrak{sl}(10)}(2,0)^e$. The locality of $\dot Y(u,w)\dot Y(v,z)$ is easy to see for either $u\in L_{\mathfrak{sl}(10)}(2,0)$ or $v\in L_{\mathfrak{sl}(10)}(2,0)$. Thus we only need to consider locality of the case $\dot Y(u,w)\dot Y(v,z)$ for both $\dot u,\dot v\in L_{\mathfrak{sl}(10)}(2,\Lambda_3+\Lambda_7)$. We focus on the four point function

$$E\langle\dot u_4,\dot J^{\dot 6}(\dot u_3,w)\dot J^{\dot 6}(\dot u_2,z)\dot u_1\rangle$$

which need to be rational and symmetric.

We first only focus on choosing $\dot u_1,\dot u_2,\dot u_3,\dot u_4$ to be primary vectors as in Lemma 3.2. In this case, we only need to check the dotted equation (3.7).

Lemma 3.4. *If one of $\dot\lambda_1,\dot\lambda_4$ is $\dot 0$, the dotted equation (3.7) is automatically satisfied (dotted (3.7) is trivial in this case).*

Proof. Suppose $\lambda_4=0,\lambda_1=6$. Then $\gamma=\mu=6$, i.e there is only one possible channel. Equation (3.7) implies

$$c^0_{6,6}c^6_{6,6}(B^{6,6}_{0,6})_{6,6}=c^0_{6,6}c^6_{6,6}.$$

Since $c^0_{6,6}\cdot c^6_{6,6}\neq 0$, one gets $(B^{6,6}_{0,6})=1$.

From equation (3.8), we have

$$(B^{6,6}_{0,6})_{6,6}(\dot B^{\dot 6,\dot 6}_{\dot 0,\dot 6})_{\dot 6,\dot 6}=1.$$

which implies $\dot B^{\dot 6,\dot 6}_{\dot 0,\dot 6}=1$, so the dotted equation (3.7) is trivial. □

It remains to deal with the case $\dot\lambda_1 = \dot\lambda_2 = \dot\lambda_3 = \dot\lambda_4 = \dot 6$. For simplicity, we write $B = B^{6,6}_{6,6}$ and $\dot B = B^{\dot 6,\dot 6}_{\dot 6,\dot 6}$.

Lemma 3.5. *For $\dot\lambda_1 = \dot\lambda_2 = \dot\lambda_3 = \dot\lambda_4 = \dot 6$, the dotted equation (3.7) holds by choosing*

$$c^{\dot\lambda_2}_{\dot 6,\dot\lambda_1} = c^{\lambda_2}_{6,\lambda_1}(T^{\lambda_2}_{6,\lambda_1})^{-1}$$

where $T^{\lambda_2}_{6,\lambda_1}$ are as in Lemma 2.7.

Proof. Lemma 2.7 asserts

$$B_{\mu,\gamma} = (T^6_{6,\mu})^{-1}(T^{\mu}_{6,6})^{-1}B_{\gamma,\mu}T^6_{6,\gamma}T^{\gamma}_{6,6}.$$

We also know $\dot B = (B^t)^{-1}$. In this case, we rewrite equation (3.7) as:

$$\Sigma_\mu c^6_{6,\mu}c^\mu_{6,6}B_{\mu,\gamma} = \Sigma_\mu c^6_{6,\mu}c^\mu_{6,6}(T^6_{6,\mu})^{-1}(T^{\mu}_{6,6})^{-1}B_{\gamma,\mu}T^6_{6,\gamma}T^{\gamma}_{6,6} = c^6_{6,\gamma}c^\gamma_{6,6}, \tag{3.12}$$

for $\mu, \gamma \in \{0, 6\}$, i.e.

$$\Sigma_\mu c^6_{6,\mu}c^\mu_{6,6}(T^6_{6,\mu})^{-1}(T^{\mu}_{6,6})^{-1}B_{\gamma,\mu} = c^6_{6,\gamma}c^\gamma_{6,6}(T^6_{6,\gamma})^{-1}(T^{\gamma}_{6,6})^{-1}.$$

Since $\dot B = (B^t)^{-1}$, obviously, the above equation implies

$$\Sigma_\mu c^6_{6,\mu}c^\mu_{6,6}(T^6_{6,\mu})^{-1}(T^{\mu}_{6,6})^{-1}\dot B_{\mu,\gamma} = c^6_{6,\gamma}c^\gamma_{6,6}(T^6_{6,\gamma})^{-1}(T^{\gamma}_{6,6})^{-1}.$$

In order to have dotted equation (3.7), one only needs to take

$$c^{\dot\lambda_2}_{\dot 6,\dot\lambda_1} = c^{\lambda_2}_{6,\lambda_1}(T^{\lambda_2}_{6,\lambda_1})^{-1}.$$

To ensure the skew-symmetry property of a vertex operator algebra, we need a normalization so that $J^{\dot 6}(u,z) = \mathcal{Y}^{\dot 6}_{\dot 6,\dot 0} + \frac{c^{\dot 6}_{\dot 6,\dot 6}}{c^{\dot 6}_{\dot 6,\dot 0}}\mathcal{Y}^{\dot 6}_{\dot 6,\dot 6} + \frac{c^{\dot 0}_{\dot 6,\dot 6}}{c^{\dot 6}_{\dot 6,\dot 0}}\mathcal{Y}^{\dot 0}_{\dot 6,\dot 6}$. □

Remark 3.6. *One can see clearly that in the construction of $L_{\mathfrak{sl}(10)}(2,0)^e$, the only nontrivial braiding matrix we use is $B^{6,6}_{6,6}$. That is, we only use the fact that $D^\mu_{6,6} \neq 0$, and $D^6_{6,\mu} \neq 0$, for $\mu = 0, 2, 4, 6, 8$ in the arguments. As we mentioned already that a proof of the fact using the language of vertex operator algebra is given in §7. Similar calculation could be done for $L_{\mathfrak{sl}(28)}(2,0)^e$.*

Remark 3.7. *By choosing $\dot J^{\dot 6} = \mathcal{Y}^{\dot 6}_{\dot 6,\dot 0} + a\frac{c^{\dot 6}_{\dot 6,\dot 6}}{c^{\dot 6}_{\dot 6,\dot 0}}\mathcal{Y}^{\dot 6}_{\dot 6,\dot 6} + a^2\frac{c^{\dot 0}_{\dot 6,\dot 6}}{c^{\dot 6}_{\dot 6,\dot 0}}\mathcal{Y}^{\dot 0}_{\dot 6,\dot 6}$ for any $a \in \mathbb{C}^*$, the dotted equation (3.7) is still satisfied. We will first prove there is a vertex operator algebra structure on $L_{\mathfrak{sl}(10)}(2,0)^e$ and then in the next section prove that different choices of the vertex operators actually give the same vertex operator algebra structure.*

Lemma 3.8. *$E\langle \dot u_4, \dot J^{\dot\lambda_3}(\dot u_3, w)\dot J^{\dot\lambda_2}(\dot u_2, z)\dot u_1\rangle$ is rational, symmetric for all higher descendants.*

Proof. Symmetric property follows from Lemma 3.2 without the assumption that $\dot u_i$ is primary for $i = 1, 2, 3, 4$.

Let $\{x_{i,j}|, i, j = 1, \cdots, 10, i \neq j\} \cup \{h_i | 1 \leq i \leq 9\}$ be the standard basis of $\mathfrak{sl}(10)$ as in [Hu]. Fix $\dot u_i$ i=1,2,3,4 as in Lemma 3.2. Assume that $n = 0$ and $i > j$, or $n > 0$ for any i, j. Using

equation (2.2), we have

$$
\begin{aligned}
(3.13)\qquad & E\langle x_{i,j}(-n)\dot{u}_4, \dot{J^{\lambda_3}}(\dot{u}_3,w)\dot{J^{\lambda_2}}(\dot{u}_2,z)\dot{u}_1\rangle\\
&= - E\langle \dot{u}_4, x_{i,j}(n)\dot{J^{\lambda_3}}(\dot{u}_3,w)\dot{J^{\lambda_2}}(\dot{u}_2,z)\dot{u}_1\rangle\\
&= - E\langle \dot{u}_4, \dot{J^{\lambda_3}}(x_{i,j}(0)\dot{u}_3,w)\dot{J^{\lambda_2}}(\dot{u}_2,z)\dot{u}_1\rangle - E\langle \dot{u}_4, \dot{J^{\lambda_3}}(\dot{u}_3,w)x_{i,j}(n)\dot{J^{\lambda_2}}(\dot{u}_2,z)\dot{u}_1\rangle\\
&= - E\langle \dot{u}_4, \dot{J^{\lambda_3}}(x_{i,j}(0)\dot{u}_3,w)\dot{J^{\lambda_2}}(\dot{u}_2,z)\dot{u}_1\rangle - E\langle \dot{u}_4, \dot{J^{\lambda_3}}(\dot{u}_3,w)\dot{J^{\lambda_2}}(x_{i,j}(0)\dot{u}_2,z)\dot{u}_1\rangle\\
&\quad - E\langle \dot{u}_4, \dot{J^{\lambda_3}}(\dot{u}_3,w)\dot{J^{\lambda_2}}\dot{u}_2,z)(x_{i,j}(n)\dot{u}_1\rangle\\
&= - E\langle \dot{u}_4, \dot{J^{\lambda_3}}(x_{i,j}(0)\dot{u}_3,w)\dot{J^{\lambda_2}}(\dot{u}_2,z)\dot{u}_1\rangle - E\langle \dot{u}_4, \dot{J^{\lambda_3}}(\dot{u}_3,w)\dot{J^{\lambda_2}}(x_{i,j}(0)\dot{u}_2,z)\dot{u}_1\rangle,
\end{aligned}
$$

which is a rational function by Lemmas 3.4 and 3.5.

Using a similar argument, one easily gets

$$\langle \dot{u}_4, \dot{J^{\lambda_3}}(\dot{u}_3,w)\dot{J^{\lambda_2}}(\dot{u}_2,z)x_{i,j}(-n)\dot{u}_1\rangle$$

is also rational for either $n=0$ and $i<j$, or $n>0$. Thus $\langle \dot{u}_4, \dot{J^{\lambda_3}}(\dot{u}_3,w)\dot{J^{\lambda_2}}(\dot{u}_2,z)\dot{u}_1\rangle$ is rational for any $\dot{u}_4, \dot{u}_1 \in L_{\mathfrak{sl}(10)}(2,0)^e$ and primary elements $\dot{u}_2$, $\dot{u}_3$. Together with the symmetric property this implies for any two primary elements $\dot{u}_2, \dot{u}_3$, there exists some $k \geq 0$, such that

$$(w-z)^k[\dot{J^{\lambda_3}}(\dot{u}_3,w), \dot{J^{\lambda_2}}(\dot{u}_2,z)] = 0$$

which is the locality condition. Since the $L_{\mathfrak{sl}(10)}(2,\dot{0})$-module $L_{\mathfrak{sl}(10)}(2,\dot{0}) \oplus L_{\mathfrak{sl}(10)}(2,\Lambda_3+\Lambda_7)$ is generated by the primary elements, it follows from Lemma 2.5 that for any $\dot{u}, \dot{v} \in L_{\mathfrak{sl}(10)}(2,\dot{0}) \oplus L_{\mathfrak{sl}(10)}(2,\Lambda_3+\Lambda_7)$ there exists $r \geq 1$ such that

$$(z_1-z_2)^r[\dot{J^{\lambda_3}}(\dot{u},z_1), \dot{J^{\lambda_2}}(\dot{v},z_2)] = 0,$$

as expected. $\square$

Theorem 3.1 now follows from Lemma 3.8.

Remark 3.9. *There have been some work in literature on finding rational solutions of the KZ-equations (cf.* [RST] *and references therein) in special cases. Lemma 3.8 indicates that there is a rational solution of the KZ-equation for $SU(10)_2$.*

4. Uniqueness of $L_{\mathfrak{sl}(10)}(2,0)^e$

We discuss the uniqueness of the vertex operator algebra structure on $L_{\mathfrak{sl}(10)}(2,0)^e$ in this section. We first prove the uniqueness of $L_{\mathfrak{sl}(2)}(10,0)^e = L_{\mathfrak{sl}(2)}(10,0) \oplus L_{\mathfrak{sl}(2)}(10,6)$.

Lemma 4.1. *Assume that $L_{\mathfrak{sl}(2)}(10,0)^e = L_{\mathfrak{sl}(2)}(10,0) \oplus L_{\mathfrak{sl}(2)}(10,6)$ is a simple vertex operator algebra which is an extension of $L_{\mathfrak{sl}(2)}(10,0)$. Let Y denote the vertex operator on the vertex operator algebra $L_{\mathfrak{sl}(2)}(10,0)^e$. As in §3, for $u \in L_{\mathfrak{sl}(2)}(10,6)$, set*

$$Y(u,z) = J^6(u,z) = \sum_{\lambda_1,\lambda_2 \in \{0,6\}} d^{\lambda_2}_{6,\lambda_1}\mathcal{Y}^{\lambda_2}_{6,\lambda_1}(u,z),$$

then

$$d^{\lambda_2}_{6,\lambda_1} \neq 0 \text{ if } \mathcal{V}^{\lambda_2}_{6,\lambda_1} \neq 0.$$

Proof. It is clear $d^6_{6,0} = c^6_{6,0} \neq 0$ by the skew symmetry. Note that $L_{\mathfrak{sl}(2)}(10,0)^e$ has a non-degenerate, symmetric, invariant bilinear form $(\cdot,\cdot)$ (cf. [Li1]). This implies that $d^0_{6,6} \neq 0$.

We now prove that $d^6_{6,6} \neq 0$. Assume that $d^6_{6,6} = 0$. It is known that the weight one subspace $\mathfrak{sl}(2) + L(6)$ of $L_{\mathfrak{sl}(2)}(10,0)^e$ is a Lie algebra denoted by $\mathfrak{g}$ where $L(6)$ is the irreducible module

with highest weight 6 for $\mathfrak{sl}(2)$. If $d_{6,6}^6 = 0$, then $[L(6), L(6)] = \mathfrak{sl}(2)$ or 0. If $[L(6), L(6)] = \mathfrak{sl}(2)$, then $\mathfrak{g}$ is a simple Lie algebra. According to the classification of finite dimensional simple Lie algebras, the only possibility for $\mathfrak{g}$ is B_2, thus $L_{\mathfrak{sl}(2)}(10,0)^e = L_{B_2}(1,0)$, a contradiction.

If $[L(6), L(6)] = 0$, then $L(6)$ generates a Heisenberg vertex operator algebra U with central charge 7. The character of the Heisenberg vertex operator algebra U is

$$\mathrm{ch}_q U = \frac{q^{-\frac{5}{48}}}{\prod_{n\geq 1}(1-q^n)^7}$$

here 5/2 is the central charge of $L_{\mathfrak{sl}(2)}(10,0)$. By applying Lemma 2.2 to $L_{\mathfrak{sl}(2)}(10,0)+L_{\mathfrak{sl}(2)}(10,6)$ as a $L_{\mathfrak{sl}(2)}(10,0)$-module, we immediately get that the coefficients of

$$\eta(q)^{5/2}\mathrm{ch}_q(L_{\mathfrak{sl}(2)}(10,0) + L_{\mathfrak{sl}(2)}(10,6))$$

satisfy the polynomial growth condition. But the coefficients of

$$\eta(q)^{5/2}\mathrm{ch}_q U = \frac{1}{\prod_{n\geq 1}(1-q^n)^{9/2}}$$

has exponential growth, a contradiction. The proof is complete. $\square$

Remark 4.2. *By Lemma 4.1 and*

$$d_{6,0}^6 d_{6,6}^0 B_{0,0} + d_{6,6}^6 d_{6,6}^6 B_{6,0} = d_{6,0}^6 d_{6,6}^0$$
$$d_{6,0}^6 d_{6,6}^0 B_{0,6} + d_{6,6}^6 d_{6,6}^6 B_{6,6} = d_{6,6}^6 d_{6,6}^6$$

which is an expansion of equation (3.12) with $c_{\lambda,\mu}^\gamma$ replaced by $d_{\lambda,\mu}^\gamma$, we see that the only option of J^6 that gives a vertex operator algebra structure on the space $L_{\mathfrak{sl}(2)}(10,0)^e$ is

$$J^6 = c_{6,0}^6\mathcal{Y}_{6,0}^6 + a^2(c_{6,6}^0\mathcal{Y}_{6,6}^0) + a(c_{6,6}^6\mathcal{Y}_{6,6,}^6), \text{ for any } a \in \mathbb{C}^*.$$

The following Theorem will help us to determine the uniqueness of the vertex operator algebra structure on both $L_{\mathfrak{sl}(2)}(10,0)^e$ and $L_{\mathfrak{sl}(10)}(2,0)^e$.

Theorem 4.3. *Let $(V, Y, \mathbf{1}, \omega)$ be a vertex operator algebra with a linear isomorphism g which preserves $\mathbf{1}$ and ω. Set $Y^g(u,z) = g^{-1}Y(gu,z)g$ for any $u \in V$. Then $(V, Y^g, \mathbf{1}, \omega)$ is a vertex operator algebra isomorphic to $(V, Y, \mathbf{1}, \omega)$.*

Proof. We first check the vertex operator algebra axioms for $(V, Y^g, \mathbf{1}, \omega)$. Since $g\mathbf{1} = \mathbf{1}$ and $g\omega = \omega$, we only need to check the creativity, derivation property and the commutativity.

1) Creativity: For $u \in V$

$$\lim_{z\to 0} Y^g(u,z)\mathbf{1} = \lim_{z\to 0} g^{-1}Y(gu,z)g\mathbf{1} = g^{-1}\lim_{z\to 0} Y(gu,z)\mathbf{1} = g^{-1}gu = u.$$

2) Derivation property: Let $Y^g(\omega,z) = \sum_{n\in\mathbb{Z}} L^g(n)z^{-n-2}$. Then

$$\begin{aligned}
[L^g(n), Y^g(u,z)] &= [g^{-1}L(-1)g, g^{-1}Y(gu,z)g] \\
&= g^{-1}[L(-1), Y(gu,z)]g \\
&= g^{-1}\frac{d}{dz}Y(gu,z)g \\
&= \frac{d}{dz}g^{-1}Y(gu,z)g \\
&= \frac{d}{dz}Y^g(u,z).
\end{aligned}$$

3) Commutativity: For any $u, z \in V$, by commutativity of $(V, Y, \mathbf{1}, \omega)$, there exists $n \in \mathbb{Z}$ such that

$$(z_1 - z_2)^n[Y(gu,z_1), Y(gv,z_2)] = 0.$$

This implies that

$$(z_1 - z_2)^n [Y^g(u, z_1), Y^g(v, z_2)] = 0.$$

Thus $(V, Y^g, \mathbf{1}, \omega)$ is a vertex operator algebra. It is clear that the linear map $g : V \to V$ gives a vertex operator algebra isomorphism from $(V, Y^g, \mathbf{1}, \omega)$ to $(V, Y, \mathbf{1}, \omega)$. $\square$

Corollary 4.4. *The simple vertex operator algebra structure on $L_{\mathfrak{sl}(2)}(10, 0)^e$ is unique.*

Proof. Let $V = L_{\mathfrak{sl}(2)}(10, 0) + L_{\mathfrak{sl}(2)}(10, 6)$ be a simple vertex operator algebra which is an extension of $L_{\mathfrak{sl}(2)}(10, 0)$. Then $J^6 = c^6_{6,0}\mathcal{Y}^6_{6,0} + a^2(c^0_{6,6}\mathcal{Y}^0_{6,6}) + a(c^6_{6,6}\mathcal{Y}^6_{6,6,})$, for some $a \in \mathbb{C}^*$. Note that $L_{B_2}(1, 0)$ and V are isomorphic $L_{\mathfrak{sl}(2)}(10, 0)$-modules. Let $g : L_{B_2}(1, 0) \to L_{B_2}(1, 0)$ be the linear map such that $g|_{L_{\mathfrak{sl}(2)}(10,0)} = 1$ and $g|_{L_{\mathfrak{sl}(2)}(10,6)} = a$. Then V and $L_{B_2}(1, 0)^g$ are isomorphic by noting that for $u \in L_{\mathfrak{sl}(2)}(10, 6)$,

$$Y^g(u, z) = c^6_{6,0}\mathcal{Y}^6_{6,0}(u, z) + ac^6_{6,6}\mathcal{Y}^6_{6,6}(u, z) + a^2c^0_{6,6}\mathcal{Y}^0_{6,6}(u, z)$$

(see Remark 4.2). Thus, V and $L_{B_2}(1, 0)$ are isomorphic by Theorem 4.3. $\square$

The following corollary follows from a similar argument.

Corollary 4.5. *The simple vertex operator algebra structure on $L_{\mathfrak{sl}(10)}(2, 0)^e$ is unique.*

5. Mirror extension of $L_{\mathfrak{sl}(28)}(2, 0)$

We give another example of mirror extension which is the extension of $L_{\mathfrak{sl}(28)}(2, 0)$ in this section. Although this example is more complicated than the example given in Section 3, the ideals and the methods are similar.

Consider the conformal inclusion $SU(2)_{28} \subset (G_2)_1$ (see [CIZ, GNO]) and the level-rank duality $SU(2)_{28} \times SU(28)_2 \subset SU(56)_1$. Due to [CIZ, DLN], one knows that the vertex operator algebra

$$L_{G_2}(1, 0) = L_{\mathfrak{sl}(2)}(28, 0) \oplus L_{\mathfrak{sl}(2)}(28, 10) \oplus L_{\mathfrak{sl}(2)}(28, 18) \oplus L_{\mathfrak{sl}(2)}(28, 28), \tag{5.1}$$

where $L_{\mathfrak{sl}(2)}(28, 0) \subset L_{G_2}(1, 0)$ is a conformal embedding with central charge $\frac{14}{5}$.

The decomposition of $L_{\mathfrak{sl}(56)}(1, 0)$ under $L_{\mathfrak{sl}(2)}(28, 0) \otimes L_{\mathfrak{sl}(28)}(2, 0)$ is given in Lemma 2.8 as follow

$$L_{\mathfrak{sl}(56)}(1, 0) = \bigoplus_{a=0,\ a \text{ even},}^{28} a \times \dot{a}, \tag{5.2}$$

where $a = L_{\mathfrak{sl}(2)}(28, a)$ and $\dot{a} = L_{\mathfrak{sl}(28)}(2, \Lambda_{\frac{a}{2}} + \Lambda_{28-\frac{a}{2}})$, here Λ_i is the fundamental weight of $\mathfrak{sl}(28)$, and we use $\Lambda_{28} = \Lambda_0$ (sometimes 0 by abusing of notations) to denote the trivial representation of $\mathfrak{sl}(28)$.

By the Mirror Extension Conjecture and equations (5.1) and (5.2), it is expected that there is a vertex operator algebra structure on

$$V = L_{\mathfrak{sl}(28)}(2, 0)^e = L_{\mathfrak{sl}(28)}(2, 0) + L_{\mathfrak{sl}(28)}(2, \Lambda_5 + \Lambda_{23}) + L_{\mathfrak{sl}(28)}(2, \Lambda_9 + \Lambda_{19}) + L_{\mathfrak{sl}(28)}(2, 2\Lambda_{14})$$

with central charge $\frac{261}{5}$. Note that the vertex operator algebra structure on V cannot be obtained from the framed vertex operator algebras as $\frac{261}{5}$ is not a half integer. For conveniece, we use $V = \dot{0} + \dot{10} + \dot{18} + \dot{28}$.

Theorem 5.1. *There is a vertex operator algebra structure on $V = L_{\mathfrak{sl}(28)}(2, 0)^e$.*

Proof. Again, we only need to define the vertex operator $\dot{Y}$ on V satisfying locality.

We introduce some notations first. We use $\tilde{Y}$, Y and $\dot{Y}$ to denote the vertex operators on $L_{\mathfrak{sl}(56)}(1,0)$, $L_{G_2}(1,0)$ and V respectively. As in §3 , for $u_1 \in L_{\mathfrak{sl}(2)}(28,\lambda)$, $u_2 \in L_{\mathfrak{sl}(28)}(2,\dot{\lambda})$,

$$\tilde{Y}(u_1 \otimes u_2, z) = \Sigma_{\lambda_1,\lambda_2} D^{\lambda_2}_{\lambda,\lambda_1} \cdot \mathcal{Y}^{\lambda_2}_{\lambda,\lambda_1}(u_1,z) \otimes \mathcal{Y}^{\dot{\lambda}_2}_{\dot{\lambda},\dot{\lambda}_1}(u_2,z), \tag{5.3}$$

$$Y(u_1,z) = \sum_{\lambda_1,\lambda_2} c^{\lambda_2}_{\lambda,\lambda_1} \mathcal{Y}^{\lambda_2}_{\lambda,\lambda_1}(u_1,z) \tag{5.4}$$

where $\mathcal{Y}^{\lambda_2}_{\lambda,\lambda_1} \in \mathcal{V}^{\lambda_2}_{\lambda,\lambda_1}$, $\mathcal{Y}^{\dot{\lambda}_2}_{\dot{\lambda},\dot{\lambda}_1} \in \mathcal{V}^{\dot{\lambda}_2}_{\dot{\lambda},\dot{\lambda}_1}$. As in equation (3.5), we can assume that $D^{\lambda_2}_{\lambda,\lambda_1} = 1$ whenever $\mathcal{V}^{\lambda_2}_{\lambda,\lambda_1} \neq 0$ for $\lambda, \lambda_1, \lambda_2 \in \{0, 10, 18, 28\}$ by suitably choosing the basis of intertwining operators $\mathcal{Y}^{\lambda_2}_{\lambda,\lambda_1}$ and $\dot{\mathcal{Y}}^{\dot{\lambda}_2}_{\dot{\lambda},\dot{\lambda}_1}$. This can also be calculated in the frame work of vertex operator algebra similarly as in §7.

We now determine the vertex operator $\dot{Y}$ on V. For $u \in L_{\mathfrak{sl}(28)}(2,\dot{\lambda})$, as in §3, we write

$$\dot{Y}(u,z) = \dot{J}^{\dot{\lambda}}(u,z) = \sum_{\dot{\lambda}_1,\dot{\lambda}_2} \dot{c}^{\dot{\lambda}_2}_{\dot{\lambda},\dot{\lambda}_1} \mathcal{Y}^{\dot{\lambda}_2}_{\dot{\lambda},\dot{\lambda}_1}.$$

We need the coefficients satisfy an equation similar to the dotted equation (3.7).

For $u \in L_{\mathfrak{sl}(28)}(2,0)$, the choice of $\dot{Y}(u,z)$ is obviously. For $u \in L_{\mathfrak{sl}(28)}(2,\Lambda_5+\Lambda_{23})$, same as in Lemma 3.5, we can take

$$\dot{c}^{\dot{\lambda}_2}_{\dot{10},\dot{\lambda}_1} = c^{\lambda_2}_{10,\lambda_1}(T^{\lambda_2}_{10,\lambda_1})^{-1}$$

to guarantee locality.

Since $L_{\mathfrak{sl}(28)}(2,2\Lambda_{14})$ is a simple current, the braiding matrix $B^{\dot{28},\dot{\lambda}_2}_{\dot{\lambda}_4,\dot{\lambda}_1}$ is just a number. Equations similar to (3.7) and (3.8) imply that

$$B^{28,\lambda_2}_{\lambda_4,\lambda_1} = (B^{\lambda_2,28}_{\lambda_4,\lambda_1})^{-1} = B^{\dot{\lambda}_2,\dot{28}}_{\dot{\lambda}_4,\dot{\lambda}_1}.$$

Using

$$c^{\lambda_4}_{\lambda_2,\mu} c^{\mu}_{28,\lambda_1} B^{\lambda_2,28}_{\lambda_4,\lambda_1} = c^{\lambda_4}_{28,\gamma} c^{\gamma}_{\lambda_2,\lambda_1}$$

and equation (2.7), we get

$$c^{\lambda_4}_{\lambda_2,\mu} c^{\mu}_{28,\lambda_1} (T^{\lambda_4}_{\lambda_2,\mu})^{-1} (T^{\mu}_{28,\lambda_1})^{-1} B^{28,\lambda_2}_{\lambda_4,\lambda_1} = c^{\lambda_4}_{28,\gamma} c^{\gamma}_{\lambda_2,\lambda_1} (T^{\lambda_4}_{28,\gamma})^{-1} (T^{\gamma}_{\lambda_2,\lambda_1})^{-1}, \tag{5.5}$$

or equivalently,

$$c^{\lambda_4}_{\lambda_2,\mu} c^{\mu}_{28,\lambda_1} (T^{\lambda_4}_{\lambda_2,\mu})^{-1} (T^{\mu}_{28,\lambda_1})^{-1} B^{\dot{\lambda}_2,\dot{28}}_{\dot{\lambda}_4,\dot{\lambda}_1} = c^{\lambda_4}_{28,\gamma} c^{\gamma}_{\lambda_2,\lambda_1} (T^{\lambda_4}_{28,\gamma})^{-1} (T^{\gamma}_{\lambda_2,\lambda_1})^{-1}.$$

As long as we choose $c^{\dot{\lambda}_1}_{\dot{28},\dot{\lambda}_3} = c^{\lambda_1}_{28,\lambda_3}(T^{\lambda_1}_{28,\lambda_3})^{-1}$, we get $\dot{J}^{\dot{28}}$, $\dot{J}^{\dot{0}}$ and $\dot{J}^{\dot{10}}$, which are pairwise mutually local.

We have defined $\dot{Y}(u,z)$ for $u \in L_{\mathfrak{sl}(28)}(2,0) \oplus L_{\mathfrak{sl}(28)}(2,\Lambda_5+\Lambda_{23}) \oplus L_{\mathfrak{sl}(28)}(2,2\Lambda_{14})$. It remains to define $\dot{Y}(u,z)$ for $u \in L_{\mathfrak{sl}(28)}(2,\Lambda_9+\Lambda_{19})$. Since the fusion product

$$L_{\mathfrak{sl}(28)}(2,\Lambda_5+\Lambda_{23}) \boxtimes L_{\mathfrak{sl}(28)}(2,2\Lambda_{14}) = L_{\mathfrak{sl}(28)}(2,\Lambda_9+\Lambda_{19}),$$

any $u \in L_{\mathfrak{sl}(28)}(2,\Lambda_9+\Lambda_{19})$ can be expressed as $u = \sum_i (a^i)_{m_i} b^i$, for some $a^i \in L_{\mathfrak{sl}(28)}(2,\Lambda_5+\Lambda_{23})$, and $b^i \in L_{\mathfrak{sl}(28)}(2,2\Lambda_{14})$. Thus for $u \in \dot{18}$ one can define

$$\dot{J}^{\dot{18}}(u,z) = \sum_i (\dot{J}^{\dot{10}}(a^i,z))_{m_i} \dot{J}^{\dot{28}}(b^i,z)$$

(see equation (2.1)). Lemma 2.5 ensures locality of $\dot{Y}$ defined on V. Thus $(V, \dot{Y}, \mathbf{1}, \omega)$ gives a vertex operator algebra structure on V. $\square$

For the uniqueness of the structure, it is quite similar to the proof of uniqueness of $L_{\mathfrak{sl}(10)}(2,0)^e$ by viewing V as an extension of the rational and C_2-cofinite vertex operator subalgebra

$$U = L_{\mathfrak{sl}(10)}(2,0) \oplus L_{\mathfrak{sl}(10)}(2, 2\Lambda_{14}).$$

Then $M = L_{\mathfrak{sl}(10)}(2, \Lambda_5 + \Lambda_{23}) \oplus L_{\mathfrak{sl}(10)}(2, \Lambda_9 + \Lambda_{19})$ is an irreducible U-module. Since the structure of U is unique and M also has a unique U-module structure, we derive $V = U + M$ has a unique vertex operator algebra structure as in Corollary 4.5.

6. Comments on general case

The general idea presented is in principle applicable to higher rank case. However, there are a number of technical problems which are not resolved in the literature. For example, for $SU(n)_k$ with $n \geq 3$, it is not clear if the braiding matrices coming from solutions of KZ equation are similar to unitary matrices.

From categorical point of view, Theorem 3.8 in [X2] can be seen as a statement about existence of commutative Frobenius algebras from given ones. For example, in the case of the key example, theorem 3.8 in [X2] says that $\dot{0} + \dot{6}$ is a commutative Frobenius algebra in the unitary tensor category associated with $SU(10)_2$. According to [HK], this is equivalent to the existence of local extensions of $SU(10)_2$ with spectrum $\dot{0} + \dot{6}$. However, to apply [HK] one must show that the unitary tensor category associated with $SU(10)_2$ from the operator algebra framework is the same as that coming from the theory of vertex operator algebra. In the case of $SU(10)_2$ one can presumably use the cohomology vanishing argument in [KL]. But this is not entirely clear, since the braiding matrix in operator algebra is automatically unitary, and we do not even know if for $SU(10)_2$ case, the braiding matrix from solutions of KZ equationis is similar to a unitary matrix.

The rationality of both $L_{\mathfrak{sl}(10)}(2,0)^e$ and $L_{\mathfrak{sl}(28)}(2,0)^e$ have not been established in this paper. They are completely rational in operator algebra framework, and in fact all its irreducible representations of $L_{\mathfrak{sl}(10)}(2,0)^e$ are listed on P. 96 of [X3], where their irreducible representations are used with simple current extensions to construct holomorphic $c = 24$ nets which corresponds to number 40 in Schelleken's list ([S]). It is worthy to point out that some holomorphic $c = 24$ vertex operator algebras including number 40 in Schelleken's list are constructed in [L, LS] by using framed vertex operator algebras . We are informed recently that the vertex operator algebra $L_{\mathfrak{sl}(10)}(2,0)^e$ can be realized as a coset vertex operator algebra in the framed holomorphic vertex operator algebra [L] corresponding to number 40 in Schelleken's list. We plan to investigate the connection of $L_{\mathfrak{sl}(10)}(2,0)^e$ with the framed vertex operator algebras further.

7. Appendix

We prove the claim made in Section 4 (see Remark 3.6) in the Appendix. That is, $D^{\mu}_{6,6} \neq 0$, $D^{6}_{6,\mu} \neq 0$ for $\mu = 0, 2, 4, 6, 8$ where $D^{\lambda_2}_{\lambda,\lambda_1}$ are defined in (3.5).

7.1. **Decomposition of $L_{\mathfrak{sl}(20)}(1,0)$ as $L_{\mathfrak{sl}(2)}(10,0) \otimes L_{\mathfrak{sl}(10)}(2,0)$-module.** Let V_L denote the lattice vertex operator algebra associated to the root lattice of $\mathfrak{sl}(20)$ ($V_L \cong L_{\mathfrak{sl}(20)}(1,0)$). We use $\{\epsilon_1, \cdots, \epsilon_{20}\}$ to denote the standard orthonormal basis of $\mathbb{R}^{20}$ with the usual inner product. Then $L = \Sigma^{20}_{i \neq j, i,j=1} \mathbb{Z}(\epsilon_i - \epsilon_j)$. Set $\alpha_i = \epsilon_i - \epsilon_{i+1}$, $1 \leq i \leq 19$. The lattice vertex operator algebra $V_L = M(1) \otimes \mathbb{C}^{\varepsilon}[L]$, where $\varepsilon : L \times L \to \langle \pm 1 \rangle$ is a 2-cocycle s.t. $\varepsilon(\alpha,\beta)\varepsilon(\beta,\alpha) = (-1)^{\alpha,\beta}$, $\forall\ \alpha, \beta \in L$. Set

$$x_{i,j} = E_{i,j} + E_{10+i,10+j},\ h_{i,j} = E_{ii} - E_{jj} + E_{10+i,10+i} - E_{10+j,10+j},$$

for $1 \leq i,j \leq 10$, $i \neq j$, where $E_{i,j}$ denotes the 20×20 matrix with 1 in the (i,j)-entry and 0 elsewhere. The bilinear form is defined as $(A,B) = \frac{1}{2}tr(AB)$, thus $(h_{i,j}, h_{i,j}) = 4$. Then

$$\Sigma_{i\neq j}\mathbb{C}x_{i,j} + \Sigma_{i\neq j}\mathbb{C}h_{i,j} \cong \mathfrak{sl}(10) \subset \mathfrak{sl}(20).$$

This gives a vertex operator algebra embedding $L_{\mathfrak{sl}(10)}(2,0) \subset V_L$. We use $\beta_i = E_{i,i} - E_{i+1,i+1} + E_{10+i,10+i} - E_{10+i+1,10+i+1}$, $1 \leq i \leq 9$, to denote a basis of the Cartan algebra of the sub Lie algebra $\mathfrak{sl}(10)$.

Set

$$\begin{aligned} e &= E_{1,11} + E_{2,12} + \cdots + E_{10,20}; \\ f &= E_{11,1} + E_{12,2} + \cdot + E_{20,10}; \\ h &= E_{1,1} + \cdots + E_{10,10} - E_{11,11} - \cdots - E_{20,20}, \end{aligned} \tag{7.1}$$

$\mathbb{C}e + \mathbb{C}f + \mathbb{C}h \cong \mathfrak{sl}(2) \subset \mathfrak{sl}(20)$. Notice that $(h,h) = 20$, one immediately see that the vertex operator algebra V_L has a vertex operator subalgebra $L_{\mathfrak{sl}(2)}(10,0)$ associated to the inclusion $\mathfrak{sl}(2) \subset \mathfrak{sl}(20)$.

It is easy to check that $L_{\mathfrak{sl}(10)}(2,0) \subset L_{\mathfrak{sl}(2)}(10,0)^c$ (the commutant of $L_{\mathfrak{sl}(2)}(10,0)$ in V_L). Thus on vertex operator algebra level, we have an inclusion $L_{\mathfrak{sl}(10)}(2,0) \otimes L_{\mathfrak{sl}(2)}(10,0) \subset V_L$. By considering the central charge,

$$\begin{aligned} c_{L_{\mathfrak{sl}(2)}(10,0)} &= \frac{30}{10+2} = 5/2, \\ c_{L_{\mathfrak{sl}(10)}(2,0)} &= \frac{2 \times 99}{10+2} = 33/2, \\ c_{V_L} &= 19 = 33/2 + 5/2, \end{aligned}$$

we get the inclusion is actually a conformal inclusion.

Equation (3.4) gives the decomposition of V_L as a $L_{\mathfrak{sl}(2)}(10,0) \otimes L_{\mathfrak{sl}(10)}(2,0)$ module :

$$\begin{aligned} V_L = {} & L_{\mathfrak{sl}(2)}(10,0) \otimes L_{\mathfrak{sl}(10)}(2,0) \\ & \oplus L_{\mathfrak{sl}(2)}(10,2) \otimes L_{\mathfrak{sl}(10)}(2,\Lambda_1 + \Lambda_9) \\ & \oplus L_{\mathfrak{sl}(2)}(10,4) \otimes L_{\mathfrak{sl}(10)}(2,\Lambda_2 + \Lambda_8) \\ & \oplus L_{\mathfrak{sl}(2)}(10,6) \otimes L_{\mathfrak{sl}(10)}(2,\Lambda_3 + \Lambda_7) \\ & \oplus L_{\mathfrak{sl}(2)}(10,8) \otimes L_{\mathfrak{sl}(10)}(2,\Lambda_4 + \Lambda_6) \\ & \oplus L_{\mathfrak{sl}(2)}(10,10) \otimes L_{\mathfrak{sl}(10)}(2,2\Lambda_5). \end{aligned} \tag{7.2}$$

We denote the decomposition as $0 \times \dot{0} + 2 \times \dot{2} + 4 \times \dot{4} + 6 \times \dot{6} + 8 \times \dot{8} + 10 \times \dot{10}$. For short we set $M^\lambda = \lambda \times \dot{\lambda}$. Note that each $M^\lambda = \oplus_{n=0}^{\infty} M^\lambda(n)$ is an irreducible highest weight module for the affine algebra $\hat{\mathfrak{sl}(2)} \times \hat{\mathfrak{sl}(10)}$-module and $M^\lambda(0) \cong L_{\mathfrak{sl}(2)}(\lambda) \otimes L_{\mathfrak{sl}(10)}(\dot{\lambda})$ is an irreducible $\mathfrak{sl}(2) \times \mathfrak{sl}(10)$-module where $L_{\mathfrak{g}}(\lambda)$ is the irreducible highest weight module for a finite dimensional simple Lie algebra $\mathfrak{g}$ with highest weight λ.

We now determine the highest (lowest, resp.) weight vectors of M^λ which are the highest (lowest, resp.) weight vectors of $\mathfrak{sl}(2) \times \mathfrak{sl}(10)$-modules of $M^\lambda(0)$.

Since

$$(V_L)_1 \cong \mathfrak{sl}(20) \supset \mathfrak{sl}(2) \oplus \mathfrak{sl}(10),$$

we also use $x_{i,j}$, $h_{i,j}$, e, f and h for elements in $(V_L)_1$. Set

$$
\begin{aligned}
v^1 &= e^{\epsilon_1-\epsilon_{20}}, \\
v^2 &= e^{\epsilon_1+\epsilon_2-\epsilon_{19}-\epsilon_{20}}, \\
v^3 &= e^{\epsilon_1+\epsilon_2+\epsilon_3-\epsilon_{18}-\epsilon_{19}-\epsilon_{20}}, \\
v^4 &= e^{\epsilon_1+\epsilon_2+\epsilon_3+\epsilon_4-\epsilon_{17}-\epsilon_{18}-\epsilon_{19}-\epsilon_{20}}, \\
v^5 &= e^{\epsilon_1+\epsilon_2+\epsilon_3+\epsilon_4+\epsilon_5-\epsilon_{16}-\epsilon_{17}-\epsilon_{18}-\epsilon_{19}-\epsilon_{20}}.
\end{aligned}
\tag{7.3}
$$

We claim that $v^i \in M^{2i}$ are the highest weight vectors. Since the proof is similar for all i we just demonstrate the proof for $i = 3$.

It is easy to see

$$h_0 v^3 = 6v^3, \ (\beta_7)_0 v^3 = (\beta_3)_0 v^3 = v^3,$$

and $(\beta_i)_0 v^3 = 0$ for $i \neq 3, 7$. Since

$$
\begin{aligned}
Y(e,z)v^3 &= \Sigma_{i=1}^{10} Y(e^{\epsilon_i-\epsilon_{10+i}}, z)v^3 \\
&= \Sigma_{i=1}^{10} E^-(\epsilon_{10+i}-\epsilon_i, z)E^+(\epsilon_{10+i}-\epsilon_i, z)e_{\epsilon_i-\epsilon_{10+i}} z^{\epsilon_i-\epsilon_{10+i}} v^3 \\
&= z\Sigma_{i=1}^{3} E^-(\epsilon_{10+i}-\epsilon_i, z)e_{\epsilon_i-\epsilon_{10+i}} v^3 + \Sigma_{i=4}^{7} E^-(\epsilon_{10+i}-\epsilon_i, z)e_{\epsilon_i-\epsilon_{10+i}} v^3 \\
&\quad + z\Sigma_{i=8}^{10} E^-(\epsilon_{10+i}-\epsilon_i, z)e_{\epsilon_i-\epsilon_{10+i}} v^3
\end{aligned}
$$

and

$$
\begin{aligned}
Y(f,z)v^3 &= \Sigma_{i=1}^{10} Y(e^{\epsilon_{10+i}-\epsilon_i}, z)v^3 \\
&= \Sigma_{i=1}^{10} E^-(\epsilon_i-\epsilon_{10+i}, z)E^+(\epsilon_i-\epsilon_{10+i}, z)e_{\epsilon_{10+i}-\epsilon_i} z^{\epsilon_i-\epsilon_{10+i}} v^3 \\
&= z^{-1}\Sigma_{i=1}^{3} E^-(\epsilon_i-\epsilon_{10+i}, z)e_{\epsilon_{10+i}-\epsilon_i} v^3 + \Sigma_{i=4}^{7} E^-(\epsilon_i-\epsilon_{10+i}, z)e_{\epsilon_{10+i}-\epsilon_i} v^3 \\
&\quad + z^{-1}\Sigma_{i=8}^{10} E^-(\epsilon_i-\epsilon_{10+i}, z)e_{\epsilon_{10+i}-\epsilon_i} v^3,
\end{aligned}
$$

we have

$$e_n v^3 = 0, \ \forall\ n \geq 0, \text{ and } f_n v^3 = 0, \ \forall\ n > 0.$$

Similarly,

$$(x_{i,i+1})_n v^3 = 0, \text{ if } n \geq 0, \text{ and } (x_{i+1,i})_n v^3 = 0, \text{ for } n > 0.$$

Thus v^3 is a highest weight vector of M^6.

Similarly,

$$
\begin{aligned}
v_1 &= e^{-\epsilon_1+\epsilon_{20}}, \\
v_2 &= e^{-\epsilon_1-\epsilon_2+\epsilon_{19}+\epsilon_{20}}, \\
v_3 &= e^{-\epsilon_1-\epsilon_2-\epsilon_3+\epsilon_{18}+\epsilon_{19}+\epsilon_{20}}, \\
v_4 &= e^{-\epsilon_1-\epsilon_2-\epsilon_3-\epsilon_4+\epsilon_{17}+\epsilon_{18}+\epsilon_{19}+\epsilon_{20}}, \\
v_5 &= e^{-\epsilon_1-\epsilon_2-\epsilon_3-\epsilon_4-\epsilon_5+\epsilon_{16}+\epsilon_{17}+\epsilon_{18}+\epsilon_{19}+\epsilon_{20}}
\end{aligned}
\tag{7.4}
$$

are the lowest weight vectors.

7.2. **Non-vanishing of $D^{\lambda_2}_{6,\lambda_1}$.** We can now prove that $D^{\mu}_{6,6} \neq 0$, for $\mu = 0, 2, 4, 6, 8$ and $D^{6}_{6,\gamma} \neq 0$ for $\gamma = 0, 2, 4, 6, 8$ (see Remark 3.6). Recall equations (7.2) and (3.5). Then the vertex operator $Y(u,z)$ for $u \in M^\lambda$ on V_L can be written as

$$Y(u,z) = \sum_{\lambda_1,\lambda_2 \in \{0,2,4,6,8,10\}} D^{\lambda_2}_{\lambda,\lambda_1} \mathcal{Y}^{\lambda_2}_{\lambda,\lambda_1}(u,z)$$

where $\mathcal{Y}_{\lambda,\lambda_1}^{\lambda_2}$ is a fixed intertwining operator of type $\begin{pmatrix} M^{\lambda_2} \\ M^{\lambda} \ \ M^{\lambda_1} \end{pmatrix}$.

Lemma 7.1. *If $D_{\lambda,\lambda_1}^{\lambda_2} \neq 0$, then $D_{\lambda_1,\lambda}^{\lambda_2} \neq 0$, $D_{\lambda,\lambda_2}^{\lambda_1} \neq 0$ and $D_{10-\lambda,10-\lambda_1}^{\lambda_2} \neq 0$.*

Proof. For $u \in M^\lambda$, we denote $Y(u,z)$ by $\mathcal{Y}^\lambda(u,z)$. Assume $D_{\lambda,\lambda_1}^{\lambda_2} \neq 0$. Using the skew-symmetry and the fact that V_L is self dual, we conclude $D_{\lambda_1,\lambda}^{\lambda_2} \neq 0$, $D_{\lambda,\lambda_2}^{\lambda_1} \neq 0$.

Since $D_{\lambda,\lambda_1}^{\lambda_2} \neq 0$, there exist $u \in M^\lambda$ and $v \in M^{\lambda_1}$ such that $u_m v$ has a nonzero projection to M^{λ_2} i.e.

$$\langle M^{\lambda_2}, \mathcal{Y}^\lambda(u,z) M^{\lambda_1}\rangle \neq 0$$

where we have used the fact that each M^λ is a self dual $L_{\mathfrak{sl}(2)}(10,0) \otimes L_{\mathfrak{sl}(10)}(2,0)$-module.

Fix nonzero homogeneous $b \in M^{10}$. Then $M^0 = \langle a_n b | a \in M^{10}, n \in \mathbb{Z}\rangle$. So we can find some $a \in M^{10}$ and $m \in \mathbb{Z}$ such that $a_m b = \mathbf{1}$. We have

$$\langle M^{\lambda_2}, \mathcal{Y}^\lambda(u,z_1)\mathcal{Y}^{10}(a,z_2)\mathcal{Y}^{10}(b,z_3) M^{\lambda_1}\rangle \neq 0.$$

Using the associativity of vertex operators we see that

$$\langle M^{\lambda_2}, \mathcal{Y}^{10-\lambda}(\mathcal{Y}^\lambda(u,z_1-z_2)a, z_2)\mathcal{Y}^{10}(b,z_3) M^{\lambda_1}\rangle \neq 0.$$

This implies $D_{10-\lambda,10-\lambda_1}^{\lambda_2} \neq 0$. $\square$

Thanks to the Lemma above, we only need to determine that $D_{6,6}^2$, $D_{6,6}^4$, $D_{6,6}^6$ and $D_{6,6}^8$ are nonzero (or $D_{6,6}^2$, $D_{4,4}^4$, $D_{4,4}^6$ and $D_{4,4}^8$ are nonzero). There are several cases.

1. $D_{6,6}^2 \neq 0$**:** We need to find $u, v \in M^6$ such that $u_m v$ has a nonzero projection to M^2. Take f as in equation (7.1), then

$$f_0 v^1 = e^{\epsilon_{11}-\epsilon_{20}} - e^{\epsilon_1-\epsilon_{10}} \in M^2.$$

We want to show $e^{\epsilon_{11}-\epsilon_{20}}$ and $e^{\epsilon_1-\epsilon_{10}}$ are in $M^6 \cdot M^6$, where $M^i \cdot M^j = \langle u_m v | u \in M^i, v \in M^j, m \in \mathbb{Z}\rangle$.

Direct calculations give

$$(x_{1,7})_0 v_3 = e^{-\epsilon_2-\epsilon_3-\epsilon_7+\epsilon_{18}+\epsilon_{19}+\epsilon_{20}} \in M^6, \tag{7.5}$$

$$(x_{7,10})_0((x_{1,7})_0 v_3)_4 v^3 = C e^{\epsilon_1-\epsilon_{10}} \in M^6 \cdot M^6,$$

for some $C \in \{\pm 1\}$. Similarly, $e^{\epsilon_{11}-\epsilon_{20}} \in M^6 \cdot M^6$. Thus $f_0 v^1 \in M^6 \cdot M^6$ and $D_{6,6}^2 \neq 0$.

Before we deal with the other cases, we need the following lemma which is immediate using the proof of $e^{\epsilon_1-\epsilon_{10}} \in M^6 \cdot M^6$.

Lemma 7.2. *Any element of type $e^{\pm\epsilon_{i_1}\pm\cdots\pm\epsilon_{i_\lambda}-\mp\epsilon_{10+j_1}\mp\cdots\mp\epsilon_{10+j_\lambda}}$ lies in $M^{2\lambda}$, for $\lambda = 0,1,2,3,4$, where i_k, j_l are distinct numbers in $\{1,\cdots,10\}$.*

2. $D_{4,4}^8 \neq 0$**:** It is easy to see

$$a = e^{\epsilon_3+\epsilon_4-\epsilon_{17}-\epsilon_{18}} \in M^4$$

by considering $(x_{3,1})_0(x_{4,2})_0(x_{7,9})_0(x_{8,10})_0 v^3$ (or Lemma 7.2). Direct calculation gives

$$a_{-1} v^2 = C_1 v^4 \in M^8,$$

for some $C_1 \in \{\pm 1\}$, which implies $D_{4,4}^8 \neq 0$.

3. $D^4_{4,4} \neq 0$**:** Notice that

$$M^4 \ni f_0 f_0 v^2 + 4v^2 = 2(e^{\epsilon_1+\epsilon_2-\epsilon_9-\epsilon_{10}} + e^{\epsilon_{11}+\epsilon_{12}-\epsilon_{19}-\epsilon_{20}} + e^{\epsilon_1-\epsilon_9+\epsilon_{12}-\epsilon_{20}} - e^{\epsilon_1-\epsilon_{10}+\epsilon_{12}-\epsilon_{19}} - e^{\epsilon_2-\epsilon_9+\epsilon_{11}-\epsilon_{20}} + e^{\epsilon_2-\epsilon_{10}+\epsilon_{11}-\epsilon_{20}}). \tag{7.6}$$

We need to show $e^{\epsilon_1+\epsilon_2-\epsilon_9-\epsilon_{10}}$, $e^{\epsilon_{11}+\epsilon_{12}-\epsilon_{19}-\epsilon_{20}}$, $e^{\epsilon_1-\epsilon_9+\epsilon_{12}-\epsilon_{20}}$, $e^{\epsilon_1-\epsilon_{10}+\epsilon_{12}-\epsilon_{19}}$, $e^{\epsilon_2-\epsilon_9+\epsilon_{11}-\epsilon_{20}}$, $e^{\epsilon_2-\epsilon_{10}+\epsilon_{11}-\epsilon_{20}} \in M^4 \cdot M^4$. We first prove that $e^{\epsilon_1+\epsilon_2-\epsilon_3-\epsilon_4}$ and $e^{\epsilon_1-\epsilon_3+\epsilon_{18}-\epsilon_{20}}$ are in $M^4 \cdot M^4$. By Lemma 7.2, we know that $a = e^{-\epsilon_2-\epsilon_3+\epsilon_{18}+\epsilon_{19}}, b = e^{-\epsilon_3-\epsilon_4+\epsilon_{19}+\epsilon_{20}}$ lie in M^4. Then

$$\begin{aligned} a_1 v^2 &= C_3 e^{\epsilon_1-\epsilon_3+\epsilon_{18}-\epsilon_{20}} \in M^4 \cdot M^4, \\ b_1 v^2 &= C_4 e^{\epsilon_1+\epsilon_2-\epsilon_3-\epsilon_4} \in M^4 \cdot M^4. \end{aligned} \tag{7.7}$$

for some $C_3, C_4 \in \{\pm 1\}$. Suitably choose some $(x_{i,j})_0$ acting on $e^{\epsilon_1+\epsilon_2-\epsilon_3-\epsilon_4}$ we can get

$$e^{\epsilon_1+\epsilon_2-\epsilon_9-\epsilon_{10}} \in M^4 \cdot M^4.$$

Similarly,

$$e^{\epsilon_{11}+\epsilon_{12}-\epsilon_{19}-\epsilon_{20}} \in M^4 \cdot M^4.$$

Suitable choosing $(x_{i,j})_0$ acting on $e^{\epsilon_1-\epsilon_3+\epsilon_{18}-\epsilon_{20}}$ asserts that

$$e^{\epsilon_1-\epsilon_9+\epsilon_{12}-\epsilon_{20}}, e^{\epsilon_1-\epsilon_{10}+\epsilon_{12}-\epsilon_{19}}, e^{\epsilon_2-\epsilon_9+\epsilon_{11}-\epsilon_{20}}, e^{\epsilon_2-\epsilon_{10}+\epsilon_{11}-\epsilon_{20}} \in M^4 \cdot M^4.$$

This implies that $f_0 f_0 v^2 + 4v^2 \in M^4 \cdot M^4$. Thus $D^4_{4,4} \neq 0$.

4. $D^6_{4,4} \neq 0$**:** By Lemma 7.2, we have $b = e^{-\epsilon_2-\epsilon_3+\epsilon_{17}+\epsilon_{18}} \in M^4$. One immediately see that

$$b_0 v^2 = C_2 e^{\epsilon_1-\epsilon_3+\epsilon_{17}+\epsilon_{18}-\epsilon_{19}-\epsilon_{20}}$$

for some $C_2 \in \{\pm 1\}$. We need to show the projection of $b_0 v^2$ to M^6 is nonzero. It suffices to show $M^0 \cdot b_0 v^2 = \langle u_n b_0 v^2 | u \in M^0, n \in \mathbb{Z}\rangle$ has a nonzero projection to M^6. Applying the operator $e_0 e_0 e_0$ to $b_0 v^2$, we get

$$(e_0)(e_0)(e_0) b_0 v^2 = C_3 e^{\epsilon_1+\epsilon_7+\epsilon_8-\epsilon_{13}-\epsilon_{19}-\epsilon_{20}} \in M^0 \cdot b_0 v^2$$

for some $C_3 \neq 0$. Since $e^{\epsilon_1+\epsilon_7+\epsilon_8-\epsilon_{13}-\epsilon_{19}-\epsilon_{20}} \in M^6$ (Lemma 7.2), the projection of $M^0 \cdot b_0 v^2$ to M^6 is nonzero, i.e. $D^6_{4,4} \neq 0$.

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